# 2P6 Solutions 2023

## SECTION A

1. (a) Taking $i$ to be the current flowing from the amplifier output through $C_2$ and using a 'virtual-earth' assumption we find

$$\begin{aligned} i &= C_2\dot{v}_0 \\ i &= -\frac{v_i}{R_1} - C_1\dot{v}_i. \end{aligned}$$

Eliminating $i$ and taking Laplace transforms with zero initial condition gives

$$C_2 s\bar{v}_0(s) = -\left(\frac{1}{R_1} + C_1 s\right)\bar{v}_i(s)$$

from which the expression for $K(s)$ follows. [6]

(b) First note that

$$L(s) = \frac{k}{s(Ts+1)^2}.$$

(i) $\omega_1 = T^{-1}$ gives $\angle L(j\omega_1) = -\pi$ rad by inspection. [3]

(ii) $\angle L(j\omega_2) = -5\pi/6$ rad requires $\tan^{-1}(\omega_2 T) = \frac{\pi}{6}$ hence

$$\omega_2 = \frac{1}{T}\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}T}.$$

[4]

(iii) The Nyquist diagram of $L(s)$ will be always to the right of the $-1$ point for closed-loop stability if we choose $k$ such that $|L(j\omega_2)| = 1$, which would also give the required phase margin of $60^o$. For this to hold:

$$k = \omega_2 \left|\left(j\frac{1}{\sqrt{3}} + 1\right)\right|^2 = \frac{4}{3\sqrt{3}T}.$$

[4]

(iv) The gain margin equals $20\log_{10}|L(j\omega_1)|^{-1}$ and we note that $|L(j\omega_1)| = k/(2\omega_1)$. Hence

$$\text{GM} = \frac{2\omega_1}{k} = \frac{3\sqrt{3}}{2} = 8.29 \text{ dB (3 s.f.)}$$

[4]

(v) The time delay should produce $\pi/6$ rad of phase lag at $\omega_2$ to make the system marginally stable, i.e. $\omega_2 D = \pi/6$ which means that

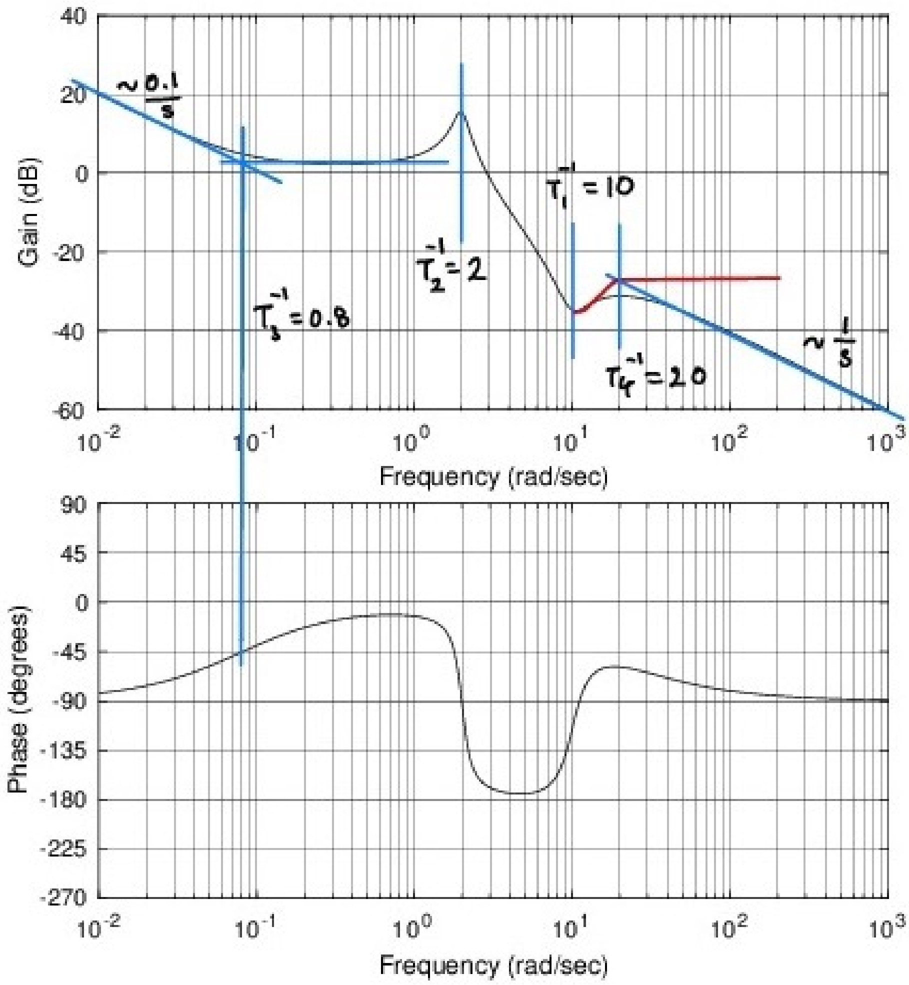
$$\frac{D}{T} = \frac{\pi}{2\sqrt{3}} = 0.907 \text{ (3 s.f.)}$$

[4]

2. (a) A straight-line asymptote at low frequencies has slope -20dB/dec and passes through $\omega = 0.1$ suggesting $a = 0.1$. The slope flattens with a break point at 0.8 rad/sec (where the phase is $-45°$) which suggests $T_3^{-1} = 0.8$. The sharp peak at 2 rad/sec suggests $T_2^{-1} = 2$ and the notch at 10 rad/sec suggests $T_1^{-1} = 10$. Without any further poles or zeros the magnitude would likely flatten as shown in red, but it starts to roll off fairly quickly suggesting another pole, i.e. $T_4^{-1} = 20$. Hence the values:

$$a = 0.1,\ T_1 = 0.1,\ T_2 = 0.5,\ T_3 = 12.5,\ T_4 = 0.05.$$

A useful check is that for large $s$, $G(s) \sim aT_1^2 T_3 T_2^{-2} T_4^{-1} s^{-1} = s^{-1}$ which agrees with the graph.

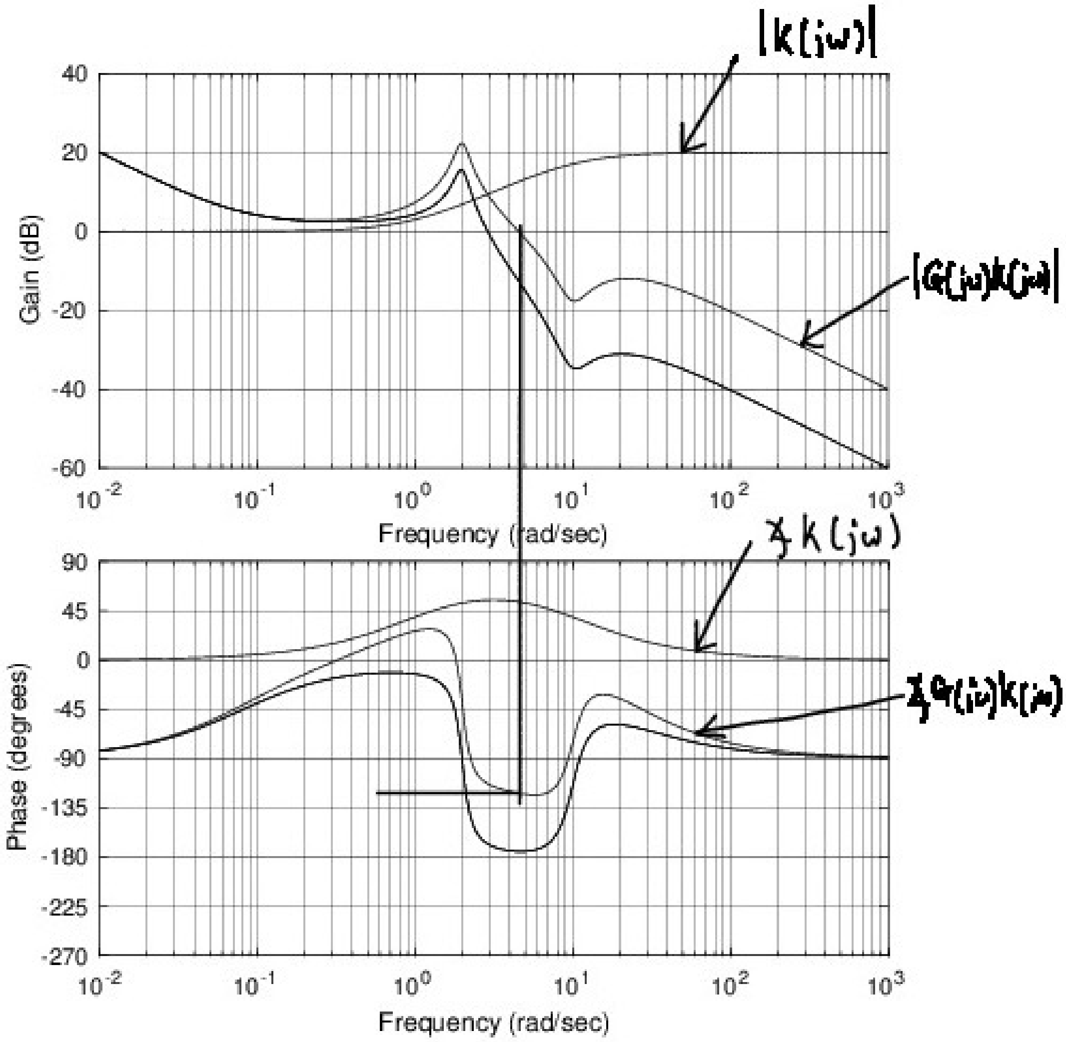



[6]

(b) (i) The system is open-loop stable and its phase is always above $-180°$ so the Nyquist diagram remains below the negative real axis, which means that the Nyquist stability criterion is satisfied for any $k$. [3]

(ii) The phase of $G(j\omega)$ equals $-135°$ at two frequencies: 2.2 rad/sec and 9.2 rad/sec. The gain at these frequencies is 12dB = 3.98 and -33dB = 0.0224. Hence the two values of $k$ are 0.251 and 44.7. [4]

(c) (i)




[5]

(ii) New gain crossover frequency $\approx 4.5$ rad/sec with phase $\approx -125°$ giving a phase margin of 55°. [Exact values: crossover = 4.53 rad, phase = $-121.5°$, PM = 58.5°.] [4]

(iii) The model being used suggests that the system will be stable with good phase margin for much increased values of loop gain. Caution is needed, however, since this is a low-order model. If there were unmodelled higher order flexible modes these could give a sharp lagging effect on the phase characteristic which might lead to instability with high gain feedback. Attention also needs to be paid to any other phase lags that could arise, e.g. due to time delays. [3]

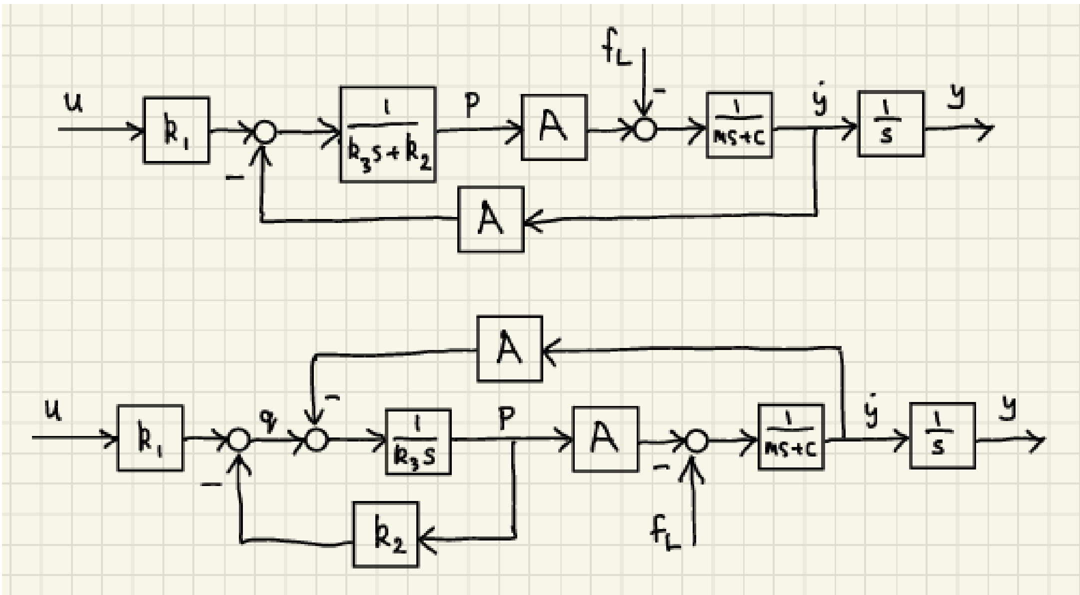
3. (a) Taking Laplace transforms and eliminating $\bar{q}$ gives the equations:

$$
\begin{aligned}
k_1\bar{u} &= As\bar{y} + (k_2 + k_3 s)\bar{p}, &(1)\\
ms^2\bar{y} + cs\bar{y} &= A\bar{p} - \bar{f}_L &(2)
\end{aligned}
$$

which gives

$$
\begin{aligned}
\bar{p} &= \frac{1}{k_2 + k_3 s}(k_1\bar{u} - As\bar{y}),\\
s\bar{y} &= \frac{1}{ms + c}(A\bar{p} - \bar{f}_L)
\end{aligned}
$$

from which the block diagram below can be drawn. The second form shows a block diagram without elimating $\bar{q}$.




[6]

(b) From the block diagram, or directly from the equations,

$$
\begin{aligned}
T_{u\to v} &= \frac{k_1 \frac{A}{(ms+c)(k_3 s+k_2)}}{1 + \frac{A^2}{(ms+c)(k_3 s+k_2)}} = \frac{k_1 A}{(ms+c)(k_3 s + k_2) + A^2},\\
T_{f_L\to v} &= \frac{\frac{-1}{ms+c}}{1 + \frac{A^2}{(ms+c)(k_3 s+k_2)}} = \frac{-(k_3 s + k_2)}{(ms+c)(k_3 s + k_2) + A^2}.
\end{aligned}
$$

These systems take the form of a proportional (negative) feedback applied to the series connection of two first-order lags. The Nyquist diagram of this series connection never crosses the negative real axis hence the system is stable. (Alternatively, the denominator is second order with all coefficients positive hence has all its roots in the left half plane.) [6]

(c) $v(t)$ settles down to a constant value in either case, after the transients have died down. Hence $y(t)$ tends to a ramp asymptotically. [4]

(d) The open loop transfer function from $u$ to $y$ looks like a positive constant times $1/s$ for small frequencies, hence its Nyquist diagram intersects the negative real axis at some finite negative value which will be to the right of the $-1$ point for sufficiently small $k$. [4]

(e) From (1)-(2), setting $s = 0$ for the steady state and substituting $u = -ky$ gives:

$$y_{ss} = -\frac{k_2}{k\,k_1 A} f_{L,ss}$$

so the steady-state gain is:

$$-\frac{k_2}{k\,k_1 A}.$$

Alternatively, writing $p(s) = s((ms + c)(k_3 s + k_2) + A^2)$ we have

$$\bar{y} = -\frac{k_3 s + k_2}{p} \bar{f}_L + \frac{k_1 A}{p} \bar{u}$$

which gives

$$(1 + \frac{k\,k_1 A}{p})\bar{y} = -\frac{k_3 s + k_2}{p} \bar{f}_L$$

and then

$$\bar{y} = -\frac{k_3 s + k_2}{k\,k_1 A + p} \bar{f}_L$$

which gives the same value for the steady-state gain on setting $s = 0$. [5]

**SECTION B**

4. (a)

$$\begin{aligned}\frac{df(t)}{dt} &= \frac{d}{dt}\frac{1}{2\pi}\int F(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int F(\omega)\frac{d}{dt}e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int j\omega F(\omega)e^{j\omega t}d\omega\end{aligned}$$

Hence $j\omega F(\omega)$ must be the FT. [5]

(b)

$$\frac{df(t)}{dt} = -2t/a^2 \exp(-t^2/a^2)$$

From the definition of $F$:

$$\begin{aligned}\frac{dF(\omega)}{d\omega} &= \frac{d}{d\omega}\int \exp(-t^2/a^2)\exp(-j\omega t)dt = \int \exp(-t^2/a^2)\frac{d}{d\omega}\exp(-j\omega t)dt \\ &= \int -jt\exp(-t^2/a^2)\exp(-j\omega t)dt = \int ja^2/2\frac{df(t)}{dt}\exp(-j\omega t)dt \\ &= -\omega a^2/2\, F(\omega)\end{aligned}$$

where the last line follows from part (a).

A solution that uses the known FT of the Gaussian to verify the result gets only partial credit. [9]

(c) Separating variables in the answer to part (b):

$$\begin{aligned}\frac{dF(\omega)}{F(\omega)} &= -\frac{a^2}{2}\omega d\omega \\ \int\frac{dF(\omega)}{F(\omega)} &= \int -\frac{a^2}{2}\omega d\omega \\ \log F(\omega) &= -\frac{a^2}{2}\omega^2/2 + C \\ F(\omega) &\propto \exp(-\frac{a^2}{4}\omega^2)\end{aligned}$$

as required.

A solution that verifies the solution by differentiation of the answer to get (b) gets partial credit. [5]

(d) Find bandwidth $B$ for 95% of energy.

Parseval shows that energy ratio is:

$$\frac{\int_{-B}^{+B}|F(\omega)^2|d\omega}{\int_{-\infty}^{+\infty}|F(\omega)^2|d\omega} = 0.95$$

Plugging in part (c)'s result with $a = 1$ (constant of prop. cancels top and bottom):

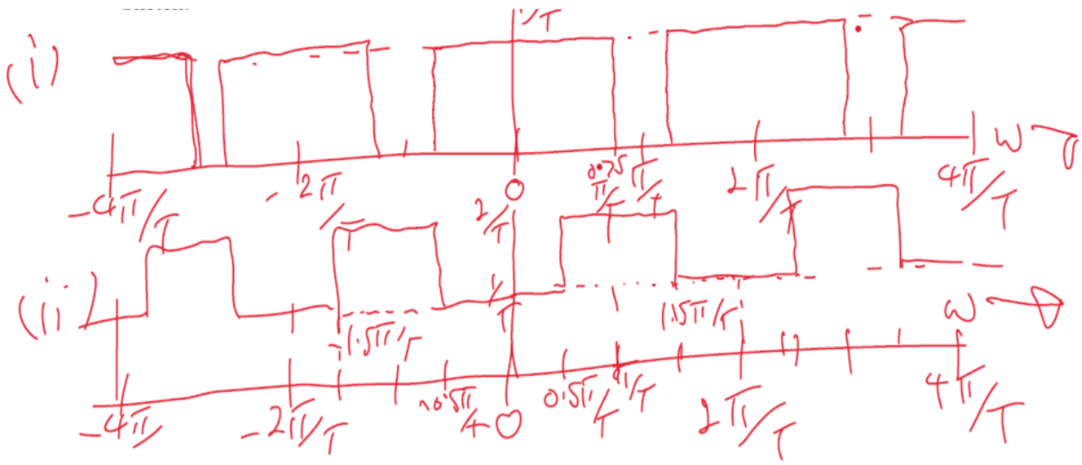
$$\frac{\frac{1}{\sqrt{2\pi}}\int_{-B}^{+B}\exp(-\omega^2/2)d\omega}{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-\omega^2/2)d\omega} = 0.95$$

But $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-\omega^2/2)d\omega = 1$ (standard Gaussian) so need

$$\frac{1}{\sqrt{2\pi}}\int_{-B}^{+B}\exp(-\omega^2/2)d\omega = \Phi(B) - (1-\Phi(B)) = 2\Phi(B) - 1 = 0.95$$

and hence $\phi(B) = 0.9525$. From tables (Maths Databook 2017 p. 29) this gives $B = 1.67$ rad/sec. [6]

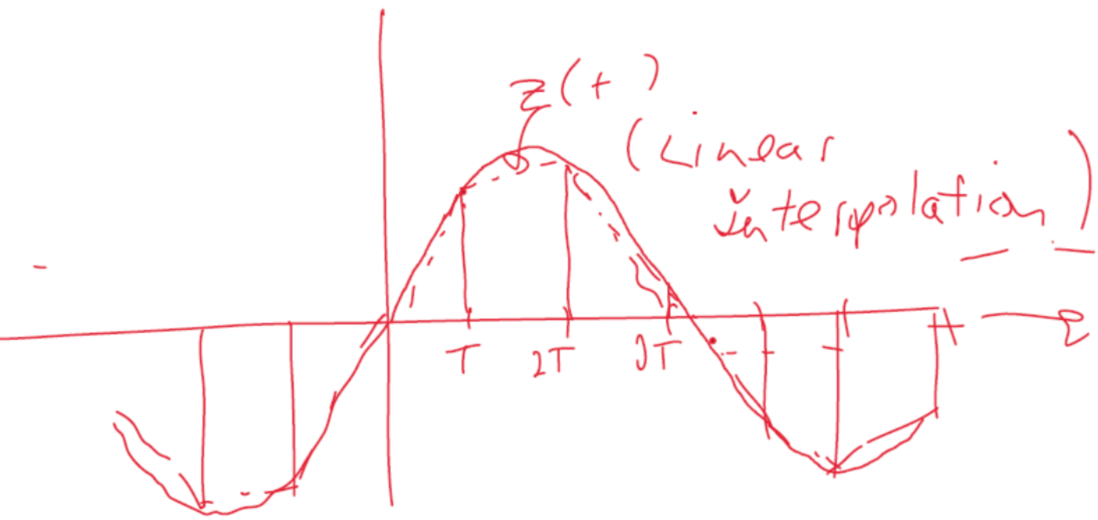
5. (a) (i) Sketch:




Nyquist sampling theory would indicate $0 < \alpha < 1$, since half of sampling frequency is $\pi/T$ rad $s^{-1}$. [6]

(ii) $z(t)$ is the linear interpolation of the digital samples.

Sketch:



Fourier transform of $\Gamma(t)$ is (Info. Data book p. 1):

$$T\text{sinc}^2(\omega T/2)$$

Sketch of this is :


Therefore sketch of spectrum of $z(t)$ is:

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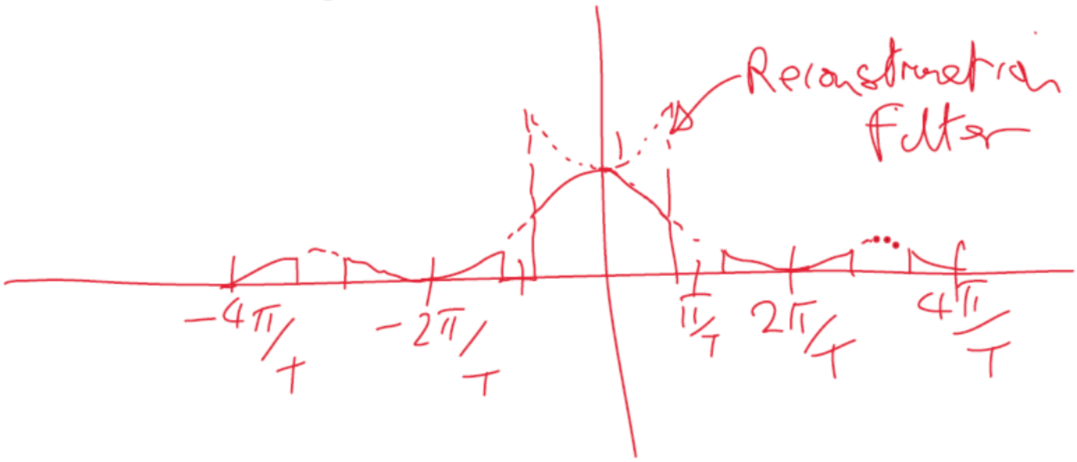
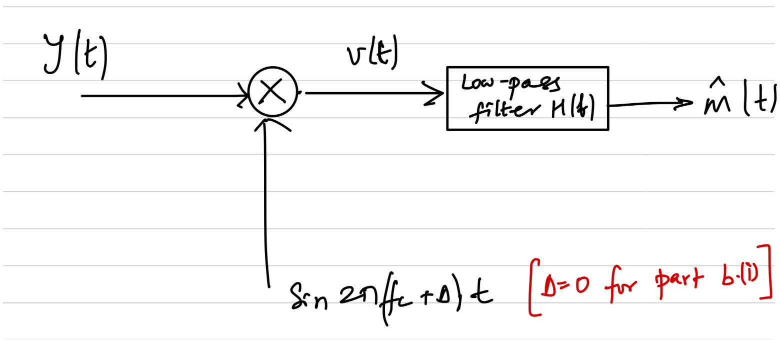



[7]

(b) (i)




The block diagram is as shown above, with $\Delta = 0$ as a perfect copy of the carrier is available. The output of the product modulator is:

$$v(t) = y(t)\sin(2\pi f_c t) = m(t)\sin^2(2\pi f_c t)$$
$$= \frac{m(t)}{2}(1 - \cos(4\pi f_c t)).$$

Therefore, the signal $m(t)$ can be recovered by applying a low-pass filter that has constant gain 2 in the band $[-W, W]$. The cut-off frequency of the filter $W_c$ can be a frequence slightly larger than $W$ such that $W_c \ll f_c$, say $W_c = 1.2W$. [6]

(ii) Now the output of the product modulator is:

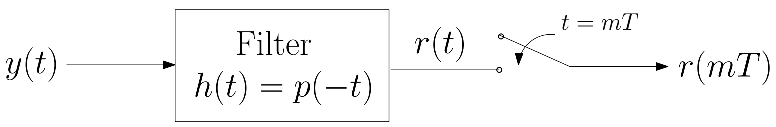
$$v(t) = y(t)\sin(2\pi(f_c + \Delta)t) = m(t)\sin(2\pi f_v t)\sin(2\pi(f_c + \Delta)t)$$
$$= \frac{m(t)}{2}[\cos(2\pi\Delta t) \; - \; \cos(2\pi(2f_c + \Delta)t)]$$

where we have used the identity $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) \; - \; \cos(A+B)]$. The term $\frac{m(t)}{2}\cos(2\pi\Delta t)$ has Fourier transform

$$\frac{1}{4}[M(f+\Delta) + M(f-\Delta)],$$

which is non-zero in the interval $[-\Delta - W, \Delta + W]$ since $M(f)$ is non-zero in $[-W, W]$. Since $\Delta = 0.1W$, this spectrum lies within the filter passband $[-1.2W, 1.2W]$. Since the filter has gain 2, its output is $m(t)\cos(2\pi\Delta t)$. [6]

6.




(a) (i) Since there is no noise, [5]

$$
\begin{aligned}
r(t) = x(t) \star p(-t) &= \int_{-\infty}^{\infty} x(\tau)p(\tau - t)d\tau \\
&= \sum_k X_k \int_{-\infty}^{\infty} p(\tau - kT)p(\tau - t)d\tau \\
&= \sum_k X_k \int_{-\infty}^{\infty} p(u + t - kT)p(u)\, du \quad (\text{using } u = \tau - t) \\
&= \sum_k X_k \int_{-\infty}^{\infty} g(t - kT) \quad (\text{using the definition of } g)
\end{aligned}
$$

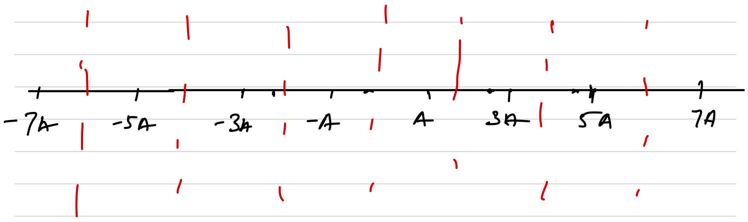
(ii) Taking $t = mT$ in part (a)(i), we have $r(mT) = X_m g(0) + \sum_{k \neq m} g((m-k)T)$. Therefore, $r(mT) = X_m$ if $g(0) = 1$ and $g((m-k)T) = 0$ for $k \neq m$. That is, [3]

$$
g(0) = 1 \quad \text{and} \quad g(nT) = 0 \text{ for } n = \pm 1, \pm 2, \ldots
$$

(iii) The pulse $p(t)$ is a baseband pulse that should be bandlimited. We would also like the energy of $p(t)$ to be largely concentrated over one symbol period (in the time domain). [3]

(b) (i) The rate of transmission is $\frac{\log_2 8}{10^{-6}} = 3 \times 10^6$ bits per second. [1]

(ii)




The decision boundaries are midway between constellation points. The overall probability of error is [9]

$$
\begin{aligned}
P_e = \frac{1}{8}\Big( &P(Y \geq -6A \mid X = -7A) + P(Y < -6A \cup Y \geq -4A \mid X = -5A) + \ldots \\
&\ldots + P(Y < 4A \cup Y \geq 6A \mid X = 5A) + P(Y < 6A \mid X = 7A)\Big) \qquad (3)
\end{aligned}
$$

The first term in (3) is

$$
\begin{aligned}
P(Y \geq -6A \mid X = -7A) &= P(-7A + N \geq -6A \mid X = -7A) \\
&= P(N > A) = \mathcal{Q}\left(\frac{A}{\sigma}\right).
\end{aligned}
$$

The second term in (3) is

$$
\begin{aligned}
&P(Y < -6A \cup Y \geq -4A \mid X = -5A) \\
&\quad = P(-5A + N < -6A \cup -5A + N \geq -4A \mid X = -5A) \\
&\quad = P(N < -A \cup N > A) \\
&\quad = 2P(N > A) = 2\mathcal{Q}\left(\frac{A}{\sigma}\right).
\end{aligned}
$$

By symmetry of the Gaussian pdf, the first and the last terms in (3) are equal, and the middle six terms are all equal to one another. Therefore,

$$P_e = \frac{1}{8}\Big(2 \cdot \mathcal{Q}\left(\frac{A}{\sigma}\right) + 6 \cdot 2\mathcal{Q}\left(\frac{A}{\sigma}\right)\Big) = \frac{7}{4}\mathcal{Q}\left(\frac{A}{\sigma}\right).$$

[11]

(iii) Using the given bound for $\mathcal{Q}(u)$, we have

$$P_e = \frac{7}{4}\mathcal{Q}\left(\frac{A}{\sigma}\right) \leq \frac{7}{8}e^{-A^2/(2\sigma^2)}.$$

As $A$ is increased, the constellation points are spaced further apart and the overall probability of error decreases exponentially with $A^2$. However the average energy used to transmit each symbol (equivalently, the average energy per bit) also increases proportionally with $A^2$. Indeed, the average energy per symbol is

$$E_s = \frac{1}{8}[2 \cdot A^2 + 2 \cdot (3A)^2 + 2 \cdot (5A)^2 + 2 \cdot (7A)^2] = 21A^2.$$

The average energy per bit is $E_b = \frac{E_s}{\log_2 8} = 7A^2$. [4]

[**Examiner's note:** Students are not expected to calculate the average energy per symbol/bit. A correct qualitative explanation of the tradeoff suffices.]

M.C. Smith, S.J. Godsill, R. Venkataramanan
24 April 2023