IB Final Exam 2016 - Crib
Section A

Q1.

(a) Irrotational/solenoidal field:

$$\nabla \cdot \mathbf{u} = 0$$

In polar coordinates:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{Pr^2}{2}\right) + \frac{\partial}{\partial z}(Qz) = -P + Q = 0$$

We can obtain the corresponding potential ϕ as

$$\nabla \phi = \mathbf{u}$$

$$\frac{\partial \phi}{\partial r} = u_r = -\frac{Pr}{2} \qquad \rightarrow \phi = -\frac{Pr^2}{4} + f(z)$$

$$\frac{\partial \phi}{\partial z} = u_z = Qz = Pz \qquad \rightarrow f' = Pz \rightarrow f = \frac{Pz^2}{2}$$

$$\phi = -\frac{Pr^2}{4} + \frac{Pz^2}{2} + \phi_0$$

The vector potential is given from the curl: $\mathbf{u} = \nabla \times \mathbf{A}$:

$$u_z = -\frac{1}{r}\frac{\partial rA}{\partial r} = Pz \qquad \rightarrow A = -\frac{Pzr}{2} + g(z)$$

$$u_r = \frac{1}{r}\frac{\partial rA}{\partial z} = -\frac{Pz}{2} \qquad \rightarrow A = -\frac{Prz}{2} + h(r)$$

Therefore we have : $\mathbf{A} = A(r,z) = -\frac{Pzr}{2}\hat{\mathbf{e}}_\theta$.

(b) The field lines are parallel to the local velocity, and are therefore given by:

$$\frac{dr}{u_r} = \frac{dz}{u_z}$$

$$\frac{dr}{-\frac{Pr}{2}} = \frac{dz}{Qz}$$

$$-\frac{2}{P}\ln r = -\frac{1}{Q}\ln z + \ln C_0$$

$$r^{2/P} z^Q = C_0$$

Note that the field lines can always be obtained, regardless of the values of P and Q.

(c) The first three integrals can be obtained by direct integration, and the fourth by using Gauss' Theorem:

(i) $\Phi_1 = \int_0^{r_o} u_z 2\pi r \, dr = Q\pi r_o^2 z_o$

(ii) $\Phi_2 = \int_0^{z_i} u_r 2\pi r_i \, dz = -\dfrac{Pr_i}{2} 2\pi r_i^2 z_i = -P\pi r_o^2 z_o$

(iii) $\Phi_2 = \int_0^{r_i} u_z 2\pi r \, dr = 0$, since $u_z(r,0) = 0$.

(iv) From Gauss' Theorem, for a volume V enclosed by surface S, $\int_S \mathbf{u} \cdot \, d\mathbf{S} = \int_V \nabla \cdot \mathbf{u} \, dV$. Therefore, $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = (Q - P)V$, so that

$$\Phi_4 = (Q - P)V - Q\pi r_o^2 z_o - (-P\pi r_o^2 z_o) = (Q - P)(V - \pi r_o^2 z_o)$$

Q2.

(a)

$$\nabla \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yzf' & fz & fy \end{vmatrix} = (f - f)\mathbf{i} + (yf' - yf')\mathbf{j} + (zf' - zf')\mathbf{k} = 0$$

So that the field is conservative. A potential can be obtained from :

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= yzf' \qquad \phi = yzf + g(y, z) \\ \frac{\partial \phi}{\partial y} &= fz = fz + \cancel{\frac{\partial g}{\partial y}} \\ \frac{\partial \phi}{\partial z} &= fy = fy + \cancel{\frac{\partial g}{\partial z}} \end{aligned}$$

Therefore, $\phi = yzf + C$. A common mistake was to take $\phi = 3yzf$.

(b)

$$\begin{aligned} \int_V \nabla \cdot \mathbf{B}\, dV &= 0 \\ \int_V yz \frac{d^2 f}{dx^2}\, dV &= 0 \end{aligned}$$

Therefore, the requirement that the integral over the relevant volume be zero means that the following conditions are possible:

(i) $\dfrac{d^2 f}{dx^2} = 0$

(ii) For the given volume, x, y and $z > 0$, so the condition can also be satisfied if $\dfrac{d^2 f}{dx^2}$ is anti-symmetric about the integration coordinate x.

(c)

$$\begin{aligned} \mathbf{C} &= \psi \mathbf{B} \\ \nabla \times \mathbf{C} &= \nabla\psi \times \mathbf{B} + \psi \cancel{\nabla \times \mathbf{B}} = 0 \\ \nabla\psi \times \nabla\phi &= 0 \qquad \textit{i.e.}\ \text{gradients must be parallel} \\ If \psi &= g(\phi) \\ \nabla\psi &= \frac{dg}{d\phi} \nabla\phi \\ \nabla\psi \times \nabla\phi &= \frac{dg}{d\phi} \nabla\phi \times \nabla\phi = 0 \end{aligned}$$

The main difficulty in the final part of this question was clearly demonstrating the last relationship.

(d) Acceptable answers in this question are either direct integration or (ideally) the recognition that the integral only depends on the end points. (i)

$$\int_C \mathbf{B}\cdot d\mathbf{l} = \phi_2 - \phi_1 = \phi(1,1,1) - \phi(-1,1,1) = (1)(1)\sin\frac{\pi}{2} - (1)(1)\sin\left(-\frac{\pi}{2}\right) = 2$$

or by direct integration

$$\int_C \mathbf{B}\cdot d\mathbf{l} = \int_{-1}^{1} yz\frac{df}{dx}\,dx = yz(f(1)-f(-1)) = (1)(1)\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right] = 2$$

The most common mistake here was a sign change, as the integral does have a direction.

(ii) For this item, direct integration is not practical, so we use the potential, or recognise that the path does not matter for the integration:

$$\int_C \mathbf{B}\cdot d\mathbf{l} = \phi_2 - \phi_1 = yz(f(1)-f(-1)) = (1)(1)\left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right] = 2$$

Since a direction was not given in the statement, both values ± 2 are acceptable answers. As the original statement of the question was ambiguous whether the same f should be used as in item (i), the answer $(f(1) - f(-1))$ was also deemed correct.

Q3.

$$y = X(x)T(t)$$
$$\frac{\partial^4 y}{\partial x^4} = X^{(}4)T$$
$$\frac{\partial^4 y}{\partial t^4} = XT''$$
$$c^2 X^{(}4)T = -T''X$$
$$c^2 \frac{X^{(}4)}{X} = -\frac{T''}{T}$$

Some may recognise the solutions to the two differential above for T and X as having eigenvalues corresponding to $\pm\omega$ and $\pm i\frac{\omega}{c}$, respectively, leading to the proposed general solutions involving sin, cos, sinh and cosh.

Alternatively, one can take the proposed solution and differentiate the terms to show that the solution is valid, *e.g.*:

$$X^{(4)} = k^4 X$$
$$T'' = -\omega^2 T$$

which can be substituted into the relationship above to yield:

$$c^2 k^4 = \omega^2$$
$$k = \sqrt{\omega/c}$$

(b) We need all three derivatives of X to apply the boundary conditions:

$$X = \quad [+A\cos kx + B\sin kx + C\cosh kx + D\sinh kx]$$
$$X' = k \quad [-A\sin kx + B\cos kx + C\sinh kx + D\cosh kx]$$
$$X'' = k^2 \quad [-A\cos kx - B\sin kx + C\cosh kx + D\sinh kx]$$
$$X''' = k^2 \quad [-A\sin kx - B\sin kx + C\cosh kx + D\sinh kx]$$

$$X(0) = A + C = 0 \rightarrow C = -A$$
$$X'(0) = B + D = 0 \rightarrow D = -B$$
$$X''(L) = k^2[-A\cos kL - B\sin kL + C\cosh kL + D\sinh kL] = 0$$
$$X'''(L) = k^2[-A\sin kL - B\sin kL + C\cosh kL + D\sinh kL] = 0$$

$$\frac{A}{B} = -\frac{\cosh kL + \cos kL}{\sinh kL - \sin kL} = -\frac{\sinh kL + \sin kL}{\cosh kL - \cos kL}$$
$$(\cosh kL + \cos kL)^2 - (\sinh kL)^2 + (\sin kL)^2 = 0$$
$$(\cosh kL)^2 + (\cos kL)^2 + 2(\cosh kL)(\cos kL) - (\sinh kL)^2 + (\sin kL)^2 = 0$$
$$2(\cosh kL)(\cos kL) = -2$$
$$(\cosh kL)(\cos kL) = -1$$

Q.E.D.

The initial condition yields:

$$T'(0) = \omega[-P\sin(\omega 0) + Q\cos(\omega 0)] = 0$$
$$Q = 0$$

so that the full general solution is represented by:

$$y(x,t) = Y_0\cos(\omega t)[\cos kx + \cosh kx + R(\sin kx + \sinh kx)]$$
$$R = [\frac{\sinh kL - \sin kL}{\cos kL + \cos kL}]$$

and kL satisfies

$\cosh kL \cos kL = -1$

1B Mathematical Methods 2016: Part B

Question 4

4a

Solve the following system of linear equations using Gaussian elimination:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 &= -1 \\ 2x_1 - 2x_2 + x_3 - 2x_4 &= 1 \\ x_1 + x_2 - 3x_3 + x_4 &= 6 \\ 3x_1 - x_2 + 2x_3 - x_4 &= 3 \end{aligned}$$

(2 marks)

Solution: First, state the equation system in matrix form:

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 2 & -2 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 & 6 \\ 3 & -1 & 2 & -1 & 3 \end{array}\right).$$

Now transform the matrix into row echelon form. Many routes are possible, for instance:

- $row_2 \leftarrow row_2 - 2 \times row_1$
- $row_3 \leftarrow row_3 - row_1$
- $row_4 \leftarrow row_4 - (row_1 + row_2)$:

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -8 & 5 & -4 & 3 \\ 0 & -2 & -1 & 0 & 7 \\ 0 & -2 & 3 & 0 & 3 \end{array}\right)$$

- $row_2 \leftarrow row_2 - 4 \times row_4$
- $row_3 \leftarrow row_3 - row_4$:

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 0 & -7 & -4 & -9 \\ 0 & 0 & -4 & 0 & 4 \\ 0 & -2 & 3 & 0 & 3 \end{array}\right)$$

- $row_2 \leftrightarrow row_4$
- $row_3 \leftarrow row_3/4$:

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -2 & 3 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -7 & -4 & -9 \end{array}\right)$$

- $row_4 \leftarrow row_4 - 7 \times row_3$:

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -2 & 3 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -4 & -16 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -16 \end{pmatrix},$$

which results in the following equations and their solutions:

$$\begin{aligned} -4x_4 = -16 &\Rightarrow x_4 = 4 \\ -x3 = 1 &\Rightarrow x_3 = -1 \\ -2x_2 + 3x_3 = 3 &\Rightarrow x_2 = -3 \\ x_1 + 3x_2 - 2x_3 + x_4 = -1 &\Rightarrow x_1 = 2 \end{aligned}$$

4b

Find the **LU** decomposition for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix}.$$

(3 marks)

Solution: **A** does not have an **LU** decomposition since $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$. However, if we swap the second and third rows in **A** we obtain:

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix},$$

with leading submatrices $\tilde{\mathbf{A}}_1 = 1$, $\tilde{\mathbf{A}}_2 = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$ and $\tilde{\mathbf{A}}_3 = \tilde{\mathbf{A}}$. Since $|\tilde{\mathbf{A}}_1| = 1$, $|\tilde{\mathbf{A}}_2| = 1$, $|\tilde{\mathbf{A}}_3| = -1$ are all nonzero, an **LU** decomposition now exists. Define:

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \mathbf{LU},$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix},$$

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix},$$

which results in:

$$\begin{aligned} U_{11} = 1, U_{12} = 2, U_{13} &= 3 \\ L_{21}U_{11} = 1 &\Rightarrow L_{21} = 1 \\ L_{21}U_{12} + U_{22} = 3 &\Rightarrow U_{22} = 1 \\ L_{21}U_{13} + U_{23} = 4 &\Rightarrow U_{23} = 1 \\ L_{31}U_{11} = 2 &\Rightarrow L_{31} = 2 \\ L_{31}U_{12} + L_{32}U_{22} = 4 &\Rightarrow L_{32} = 0 \\ L_{31}U_{13} + L_{32}U_{23} + U_{33} = 5 &\Rightarrow U_{33} = -1. \end{aligned}$$

$$\therefore \tilde{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Alternatively, with pivoting $\mathbf{A} = \mathbf{PLU}$, $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

4c

Solve

$$(\mathbf{X} - \mathbf{B})^{-1} = \mathbf{C},$$

in which

$$\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}.$$

(4 marks)

Solution:

$$\begin{aligned}(\mathbf{X} - \mathbf{B}) &= \mathbf{C}^{-1} \\ \mathbf{X} &= \mathbf{C}^{-1} + \mathbf{B}.\end{aligned}$$

Since $|\mathbf{C}| \neq 0$:

$$\begin{aligned}\mathbf{C}^{-1} &= \frac{1}{|\mathbf{C}|}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} \\ \mathbf{X} &= \mathbf{C}^{-1} + \mathbf{B} \\ &= \frac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 1.4 & 2.8 \\ 3.3 & 9.1 \end{pmatrix}.\end{aligned}$$

4d

Consider the matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 3 & a \\ 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{pmatrix}.$$

Determine a basis for the null space of $\mathbf{D}$ and state the dimensions of the null space. (8 marks)

Solution: To find the basis vectors for the null space we solve $\mathbf{Dx} = \mathbf{0}$. First, transform $\mathbf{D}$ into row echelon form with as few variables as possible:

$$\begin{aligned}\tilde{\mathbf{D}} &= \begin{pmatrix} 1 & 1 & 3 & a \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \tilde{\mathbf{D}} &= \begin{pmatrix} 1 & 1 & 3 & a \\ 0 & 1 & -2 & 1-a \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \tilde{\mathbf{D}} &= \begin{pmatrix} 1 & 0 & 5 & 2a-1 \\ 0 & 1 & -2 & 1-a \\ 0 & 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

which results in the following two equations:

$$\begin{aligned}x_1 + 5x_3 + (2a-1)x_4 &= 0 \\ x_2 - 2x_3 + (1-a)x_4 &= 0.\end{aligned}$$

We have four variables but only two equations. Define $x_3 = s$ and $x_4 = t$:

$$x_1 + 5s + (2a-1)t = 0 \Rightarrow x_1 = -5s - (2a-1)t$$
$$x_2 - 2s + (1-a)t = 0 \Rightarrow x_2 = 2s - (1-a)t.$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5s - (2a-1)t \\ 2s - (1-a)t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1-2a \\ a-1 \\ 0 \\ 1 \end{pmatrix}.$$

$\therefore$ a basis for the null space is $\begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1-2a \\ a-1 \\ 0 \\ 1 \end{pmatrix}$. Since two vectors form the basis for the null space, the dimensions of the null space is 2.

4e

If $\mathbf{v}$ is an eigenvector of an invertible matrix $\mathbf{E}$, prove that $\mathbf{v}$ is also an eigenvector of $\mathbf{E}^2$ and $\mathbf{E}^{-2}$ and state the corresponding eigenvalues. (8 marks)

Solution: Assume $\mathbf{E}\mathbf{v} = \lambda\mathbf{v}$, then:

$$\begin{aligned} \mathbf{E}^2\mathbf{v} &= \mathbf{E}(\mathbf{E}\mathbf{v}) \\ &= \mathbf{E}(\lambda\mathbf{v}) \\ &= \lambda\mathbf{E}\mathbf{v} \\ &= \lambda^2\mathbf{v}. \end{aligned}$$

$\therefore$ $\mathbf{v}$ is an eigenvector of $\mathbf{E}^2$ and the eigenvalue is λ^2.
Now, since $\mathbf{E}$ is invertible:

$$\mathbf{E}^2\mathbf{v} = \lambda^2\mathbf{v}$$
$$\Rightarrow \lambda^{-2}\mathbf{v} = \mathbf{E}^{-2}\mathbf{v}.$$

$\therefore$ $\mathbf{v}$ is an eigenvector of $\mathbf{E}^{-2}$ and the eigenvalue is λ^{-2}.

Question 5

5a

Prove that the following matrix $\mathbf{A}$ is skew-symmetric:

$$\mathbf{A} = \begin{pmatrix} 0 & 6 & -3 \\ -6 & 0 & -8 \\ 3 & 8 & 0 \end{pmatrix}.$$

(2 marks)

Solution:

$$\mathbf{A}^T = \begin{pmatrix} 0 & -6 & 3 \\ 6 & 0 & 8 \\ -3 & -8 & 0 \end{pmatrix} = -\mathbf{A}.$$

5b

Find the eigenvectors of the system

$$\mathbf{Bx} = \lambda\mathbf{x},$$

in which

$$\mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ -3 & -2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(5 marks)

Solution:

$$\mathbf{B} - \lambda\mathbf{I} = \begin{pmatrix} 1-\lambda & -1 & 2 \\ -3 & -2-\lambda & 3 \\ 2 & -1 & 1-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -3 & -2-\lambda & 3 \\ 2 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -2-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -3 & 3 \\ 2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} -3 & -2-\lambda \\ 2 & -1 \end{vmatrix}$$

$$= (1-\lambda)[(-2-\lambda)(1-\lambda)+3] + [-3(1-\lambda)-6] + 2[3-(-2-\lambda)(2)]$$

$$\Rightarrow -\lambda^3 + 7\lambda + 6 = 0$$

$$\Rightarrow \lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 3.$$

Solving for $\lambda_1 = -2$ yields $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$, $\lambda_2 = -1$ yields $\begin{pmatrix} 1 \\ 12 \\ 5 \end{pmatrix}$ and $\lambda_3 = 3$ yields $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$\therefore$ the eigenvectors are $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 12 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

5c

5c.i

Prove that the matrices $\mathbf{CC}^T$ and $\mathbf{C}^T\mathbf{C}$ have the same eigenvalues, except for the eigenvalue of 0. (6 marks)

Solution: Assume λ is a non-zero eigenvalue of $\mathbf{CC}^T$ and $\mathbf{v}$ is its corresponding eigenvector. Then $(\mathbf{CC}^T)\mathbf{v} = \lambda\mathbf{v}$. If we premultiply both sides by $\mathbf{C}^T$ we obtain $\mathbf{C}^T(\mathbf{CC}^T)\mathbf{v} = (\mathbf{C}^T\mathbf{C}) \cdot (\mathbf{C}^T\mathbf{v}) = \lambda \cdot (\mathbf{C}^T\mathbf{v})$.
$\therefore$ λ is an eigenvalue of $\mathbf{C}^T\mathbf{C}$ (and the corresponding eigenvector is $\mathbf{C}^T\mathbf{v}$).

5c.ii

Let

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

Find all eigenvalues of $\mathbf{C}^T\mathbf{C}$. (4 marks)

Solution: Since $\mathbf{C}^T\mathbf{C}$ is rather large and we know from the previous part of the subquestion that $\mathbf{CC}^T$ and $\mathbf{C}^T\mathbf{C}$ have the same eigenvalues, except for the eigenvalue of 0, we find all eigenvalues for the matrix $\mathbf{CC}^T$ instead.

$$\mathbf{CC}^T = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix},$$

with the characteristic polynomial $\lambda^2 - 8\lambda + 12$. The eigenvalues of $\mathbf{CC}^T$ are thus $\lambda_1 = 2$ and $\lambda_2 = 6$. The non-zero eigenvalues of $\mathbf{C}^T\mathbf{C}$ are therefore $\lambda_1 = 2$ and $\lambda_2 = 6$ and the remaining eigenvalues must therefore be 0 ($\lambda_3 = 0$ with algebraic multiplicity 4).

5d

Let

$$\mathbf{E} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & 5 \end{pmatrix}.$$

Find an invertible matrix $\mathbf{Z}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{Z}^{-1}\mathbf{EZ} = \mathbf{D}.$$

(8 marks)

Solution: To diagnolise $\mathbf{E}$ we first find the eigenvalues of $\mathbf{E}$:

$$
\begin{aligned}
|\mathbf{E} = \lambda \mathbf{I}| &= 0 \\
\begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 1-\lambda & 3 \\ 1 & -1 & 5-\lambda \end{vmatrix} &= (1-\lambda)[(1-\lambda)(5-\lambda)+3] - 2[(5-\lambda)-3] + 2[-1-(1-\lambda)] \\
&= \lambda(\lambda^2 - 7\lambda + 10) \\
&= 0 \\
&\Rightarrow \lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 2.
\end{aligned}
$$

Since all eigenvalues are unique, $\mathbf{E}$ can be diagonalised:

$$
\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
$$

To find the diagonalised matrix $\mathbf{Z}$ we need to find the eigenvectors for the corresponding eigenvalues:

$$
(\mathbf{E} - \lambda_i \mathbf{I})\mathbf{x}_i = 0,
$$

where $\mathbf{x}_i$ is the eigenvector for eigenvalue λ_i.
Inserting $\lambda_i = 0$ yields:

$$
(\mathbf{E} - 0\mathbf{I})\mathbf{x}_i = \mathbf{E}\mathbf{x}_i = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\Rightarrow \mathbf{x}_1 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}.
$$

Using the same strategy, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}_3 = \begin{pmatrix} -8 \\ -5 \\ 1 \end{pmatrix}$.
The eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ form the columns of $\mathbf{Z}$:

$$
\mathbf{Z} = \begin{pmatrix} -4 & 1 & -8 \\ 1 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}.
$$

Question 6

6a

Suppose x is a continuous random variable with probability density function $f(x) = \frac{1}{2\sqrt{x}}; 1 \leq x \leq 4$. Show that $f(x)$ is a suitable function for a probability density function and calculate the expected value and variance of x. (6 marks)

Solution: In order for x to be a suitable probability density function, its area should be 1:

$$\int_1^4 f(x)dx = 1$$

$$\int_1^4 f(x)dx = 1 = \int_1^4 \frac{1}{2\sqrt{x}}dx = \left[\sqrt{x}\right]_1^4 = 1$$

$\therefore f(x)$ is a suitable probability density function.
The expected value is:

$$\mathrm{E}(x) = \int_1^4 x\frac{1}{2\sqrt{x}}dx = \frac{7}{3}(\approx 2.33).$$

The variance is:

$$\mathrm{Var}(x) = \mathrm{E}[x^2] - (\mathrm{E}[x])^2 = \frac{31}{5} - \left(\frac{7}{3}\right)^2 = \frac{34}{45}(\approx 0.756).$$

6b

The breakdown of a conveyor belt is modelled by an exponential distribution with a mean of 25 days. Calculate the probability that the conveyor belt breaks down in a 40 day period. (4 marks)

Solution: The probability density function is $f(x) = \frac{1}{25}e^{-\frac{x}{25}}; x \geq 0$:

$$\begin{aligned}
\mathrm{Prob}(0 \leq x \leq 40) &= \int_0^{40} f(x)dx \\
&= \int_0^{40} \frac{1}{25}e^{-\frac{x}{25}}dx \\
&= \left[-e^{-\frac{x}{25}}\right]_0^{40} \\
&= -e^{-\frac{40}{25}} + 1 \\
&\approx 0.798.
\end{aligned}$$

6c

Assume a failure can always be repaired and repair time can be neglected. Let n be the number of failures of the product over time interval t. From first principles, derive an expression for the probability of a product failing one or more times in time interval t. (10 marks)

Solution: Subdivide t into k subintervals of length Δt: $k\Delta t = t$. Assume each subinterval is so small that only one failure can occur within it. Since repair time is neglected there can always be a failure in a subinterval. The probability of failure is then:

$$\text{Prob(failure)} = \frac{n}{k}.$$

The probability of a non-failure is:

$$\text{Prob(non-failure)} = 1 - \text{Prob(failure)} = 1 - \frac{n}{k} = \frac{k-n}{k}.$$

Since the product progresses through a series of successive subintervals Δt it is undergoing a series of trials of which the outcome is either failure or non-failure.

$\therefore$ Prob(non-failure in t) $= \left(\frac{k-n}{k}\right)^k$, and as $\Delta t \to \infty$: $\lim_{k\to\infty} \left(\frac{k-n}{k}\right)^k$.

The binomial expansion of $\left(\frac{k-n}{k}\right)^k$

$$\begin{aligned}
&= \left(1 - \frac{n}{k}\right)^k \\
&= 1 + k\left(-\frac{n}{k}\right) + \frac{k(k-1)}{2!}\left(-\frac{n}{k}\right)^2 + \frac{k(k-1)(k-2)}{3!}\left(-\frac{n}{k}\right)^3 + \dots \\
&= 1 - n + \left(\frac{k-1}{k}\right)\frac{n^2}{2!} - \left(\frac{k-1}{k}\right)\left(\frac{k-2}{k}\right)\frac{n^3}{3!} + \dots \\
&= 1 - n + \left(1 - \frac{1}{k}\right)\frac{n^2}{2!} - \left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right)\frac{n^3}{3!} + \dots
\end{aligned}$$

Let $k \to \infty$:

$$\lim_{k\to\infty} \left(\frac{k-n}{k}\right)^k = 1 - n + \frac{n^2}{2!} - \frac{n^3}{3!} + \dots,$$

which is the power series expansion of e^{-n}.

$\therefore \lim_{k\to\infty} \left(\frac{k-n}{k}\right)^k = e^{-n}$.

$\therefore$ Prob(non-failure in t) $= e^{-n}$.

$\therefore$ Prob(failure in t) $= 1 - e^{-n}$.

6d

The product in (c) consists of 8 individual parts that all must work for the product to function. The probability of failure of a part is uniformly distributed and is independent of the probability of failure of the other parts. As in (c), assume a failure can always be repaired and repair time can be neglected. On average each part fails once every 4 years. Calculate the probability of the product failing as a result of a part failing within a time period of 1.5 years. (5 marks)

Solution: Since the failure rate r is constant, $r = \frac{n}{t}$. The mean failure rate is $\frac{1}{4}$ for each part per year. 8 individual parts thus result in a failure rate $r = \frac{8}{4} = 2$ per year. From the previous subquestion we know:

$$\begin{aligned} \text{Prob(failure in } t) &= 1 - e^{-n} \\ \Rightarrow \text{Prob(failure in } t) &= 1 - e^{-rt} \end{aligned}$$

Since $r = 2$ and $t = 1.5$:

$$\begin{aligned} \text{Prob(failure in } t) &= 1 - e^{-rt} \\ &= 1 - e^{-3} \approx 0.95 \end{aligned}$$

Alternatively, a non-failure can be viewed as a Poisson process with $\lambda = 3$ and $k = 0$, thus:

$$\text{Prob(failure in } t) = 1 - \frac{\lambda^k e^{-\lambda}}{k!} = 1 - \frac{3^0 e^{-3}}{0!} = 1 - \frac{1 \times e^{-3}}{1} = 1 - e^{-3} \approx 0.95$$