EGT1 - Paper 7

Mathematical methods

1. $F = (2xz^3 + 6y)i + (6x - 2yz)j + (3x^2z^2 - y^2)k$

a) $\nabla \cdot F = \cancel{6xz^2}$
$2z^3 - 2z + 6x^2 z$

$\nabla \cdot F = 2z(3x^2 + z^2 - 1)$

@ $(1,1,0)$ $\quad \nabla \cdot F = 0$

@ $\left(\sqrt{2/3}, 1, 1\right)$ $\quad \nabla \cdot F = 2(2 + 1 - 1) = 4$

b) $$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xz^3 + 6y & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix}$$

$= (-2y + 2y)i - j(6xz^2 - 6xz^2) + k(6 - 6) = 0$

$\nabla \times F = 0 \Rightarrow$ F is Conservative field.

(c) Yes, F has Scalar potential field

$F = \nabla \phi \Rightarrow \dfrac{\partial \phi}{\partial x} = 2xz^3 + 6y \Rightarrow \phi = x^2z^3 + 6xy + f_1(y,z)$

$\dfrac{\partial \phi}{\partial y} = 6x - 2yz \Rightarrow \phi = 6xy - y^2 z + f_2(x,z)$

$\dfrac{\partial \phi}{\partial z} = (3x^2z^2 - y^2) \Rightarrow \phi = x^2z^3 - y^2 z + f_3(x,y)$

$\Rightarrow \boxed{\phi(x,y,z) = x^2z^3 + 6xy - y^2 z + C}$

(d) $\int F \cdot dr = \phi \Big|_{(1,-1,1)}^{(2,1,-1)}$ $\qquad \phi(x,y,z) = x^2 z^3 + 6xy - y^2 z + c$

$$= \phi(2,1,-1) - \phi(1,-1,1)$$

$$= (-4 + 12 + 1) - (1 - 6 - 1) = \underline{\underline{15}}$$

$$\int_{(1,-1,1)}^{(2,-1,1)} F \cdot dr = \int_1^2 (2x - 6)\, dx = (x^2 - 6x)\Big|_1^2 = -3$$

$$\int_{(2,-1,1)}^{(2,1,1)} F \cdot dr = \int_{-1}^{1} (12 - 2y)\, dy = (12y - y^2)\Big|_{-1}^{1} = 24$$

$$\int_{(2,1,1)}^{(2,1,-1)} F \cdot dr = \int_1^{-1} (12z^2 - 1)\, dz = (4z^3 - z)\Big|_1^{-1} = -6$$

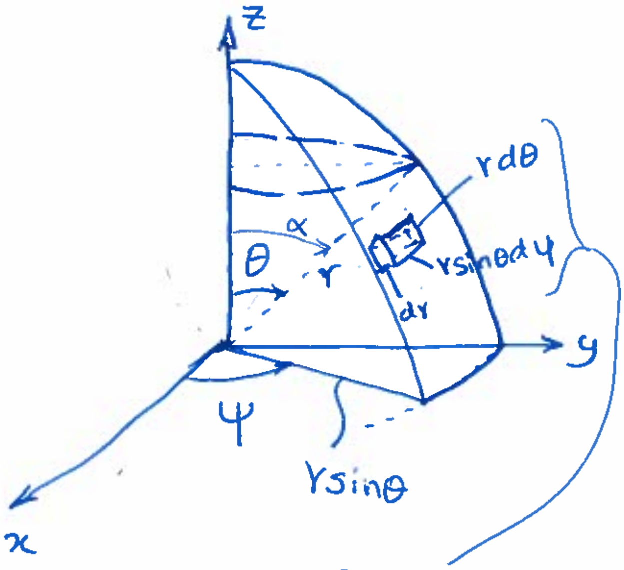
$$\underline{15}$$

$\therefore$ $\int F \cdot dl$ is independent of path. because F is a conservative field.

(e) Physically $\int_c F \cdot dr$ is the workdone in moving an object from $(1,-1,1)$ to $(2,1,-1)$ along c, which should be independent of the path in a conservative force field.

2) Sphere : $x^2+y^2+z^2 = a^2$

Cone : $z^2\sin^2\alpha = (x^2+y^2)\cos^2\alpha \qquad 0\le\alpha\le\pi$



(a) $x = r\sin\theta\cos\psi$

$y = r\sin\theta\sin\psi$

$z = r\cos\theta$

$dV = r^2\sin\theta\, d\theta\, d\psi\, dr$

Substituting these into the equation for Cone:

$$\cancel{r^2}\cos^2\theta\sin^2\alpha = \cancel{r^2}\sin^2\theta\cos^2\alpha$$

$$\Rightarrow \tan^2\alpha = \tan^2\theta \Rightarrow \tan\theta = \pm\tan\alpha.$$

$$\Rightarrow \theta = \alpha \text{ or } \pi-\alpha. \text{ (or } -\alpha)$$

Thus, it is adequate to consider $\theta = \alpha$. in spherical polar because $0\le\psi\le 2\pi$.
(See the fig. above)

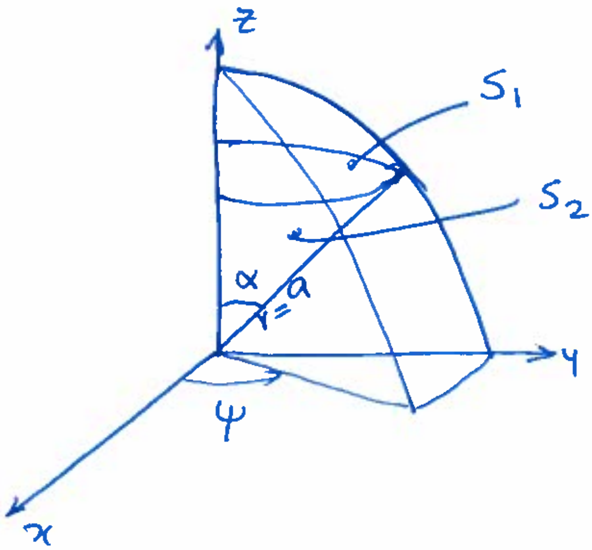
(b) $$V = \int_0^{2\pi} d\psi \int_0^{\alpha} d\theta \int_0^{a} r^2\sin\theta\, dr$$

$$= 2\pi\frac{a^3}{3}\int_0^{\alpha}\sin\theta\, d\theta = \frac{2\pi a^3}{3}(1-\cos\alpha)$$

$$\therefore \boxed{V = \frac{2\pi a^3}{3}(1-\cos\alpha)}$$

(c) Cone becomes Sphere when $\alpha = \pi$

$$\Rightarrow V = \frac{2\pi a^3}{3}(1-\cos\pi) = \frac{4\pi}{3}a^3 \quad \left(\text{a well known quantity}\right)$$



(d) $I = \left(\frac{Q}{r}\right) e_r$

$$\text{Flux} = \int_{S_1+S_2} I \cdot dA$$

if we use Gauss theorem

$$\int I \cdot dA = \int_V \nabla \cdot I \, dV$$

For spherically symmetric system $\nabla \cdot I = \frac{1}{r^2}\frac{d}{dr}(r^2 I)$

$$\Rightarrow \nabla \cdot I = \frac{Q}{r^2}\frac{d}{dr}\left(r^2/r\right) = \frac{Q}{r^2}$$

$$\int_V \nabla \cdot I = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \frac{Q}{\cancel{r^2}} \cancel{r^2} \sin\theta \, dr\, d\theta\, d\psi$$

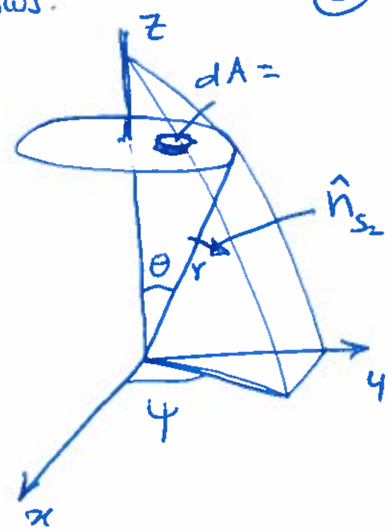
$$= Q\, 2\pi a \int_0^{\alpha} \sin\theta \, d\theta = 2\pi a Q (1-\cos\alpha)$$

$\therefore$ Flux through $(S_1 + S_2)$ is $\underline{\underline{2\pi a Q(1-\cos\alpha)}}$

(e) Since I is along e_r, flux through S_2 must be zero.

$\Rightarrow$ Flux through S_1 is $2\pi a Q (1-\cos\alpha)$

This can also be verified as follows. ⑤



$$dA = (r\sin\theta\, d\psi)(r\, d\theta)$$
$$= r^2 \sin\theta\, d\theta\, d\psi$$

$$\int I \cdot dA = Q \int_A \frac{dA}{r}$$

$$= Q \int_A r \sin\theta\, d\theta\, d\psi \quad ; \; r = a$$

$$= Q a \int_0^{2\pi} d\psi \int_0^{\alpha} \sin\theta\, d\theta = 2\pi a Q (1 - \cos\alpha)$$

This is the same as in part (d)

$\Rightarrow$ flux through S_2 must be zero.

This is because the surface unit vector for S_2 is $\perp r$ to e_r (or I).

3) $$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \qquad \alpha = \text{const.}$$

(a) $\frac{\partial \theta}{\partial x} = 0$ @ $x = 0$ & L because the ends are insulated.
$\Rightarrow$ Heat flux $q \sim \frac{\partial \theta}{\partial x}$ is zero.

(b) $\theta(x,t) = T(t)\, X(x)$

$$\Rightarrow \frac{\dot{T}}{\alpha T} = \frac{X''}{X} = -\lambda^2 \Rightarrow \begin{cases} X'' + \lambda^2 X = 0 & X'(0) = 0 \\ & X'(L) = 0 \\ \dot{T} + \lambda^2 \alpha T = 0 & T(0) = ? \end{cases}$$

for $\underline{X(x)}$: Characteristic Equation is

$$m^2 = -\lambda^2 \Rightarrow m = \pm i\lambda$$

$$\therefore X(x) = A \cos \lambda x + B \sin \lambda x$$

$$X'(x) = -A\lambda \sin \lambda x + B\lambda \cos \lambda x$$

$$X'(0) = 0 \Rightarrow B = 0.$$

$$X'(L) = 0 \Rightarrow -A\lambda \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L};\ n = 0, 1, 2, \dots$$

$$\therefore X(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x$$

for $T(t)$: $$\dot{T} = -\left(\frac{n\pi}{L}\right)^2 \alpha T$$

$$\Rightarrow T(t) = B \exp\left[-\left(\frac{n\pi}{L}\right)^2 \alpha t\right]$$ B is given by $T(0)$

$$\therefore \boxed{\theta(x,t) = \sum_{n=0}^{\infty} C_n \exp\left[-\left(\frac{n\pi}{L}\right)^2 \alpha t\right] \cos\left(\frac{n\pi}{L}\right) x}$$

(c) for $t=0$ $\quad \theta(x,0) = \theta(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{L}\right)x$

$\Rightarrow$ ~~expanding~~ writting $\theta(x)$ as Cosine Fourier Series

would give C_n

$$\therefore \quad C_0 = \frac{1}{L}\int_0^L \theta(x)\, dx$$

0.5 $$C_n = \frac{1}{L}\int_0^L \theta(x) \cos\left(\frac{n\pi}{L}\right)x\, dx \qquad n = 1, 2, \ldots$$

(d) $$\theta(x,0) = \begin{cases} \theta_0 & 0 \le x \le 0.1L \\ \theta_0\, e^{-50(x - L/10)} & 0.1L \le x \le L \end{cases}$$

~~$C_0 =$~~

$$C_0 = \frac{1}{L}\int_0^{0.1L} \theta_0\, dx + \frac{1}{L}\int_{0.1L}^{L} \theta_0\, e^{-50(x-L/10)}\, dx$$

$$= \frac{\theta_0}{10} - \frac{\theta_0}{50L}\, e^{-50(x - L/10)}\Big|_{0.1L}^{L}$$

$$= \frac{\theta_0}{10} - \frac{\theta_0}{50L}\left(e^{-45L} - 1\right)$$

$$\boxed{C_0 = \frac{\theta_0\left(5L + 1 - e^{-45L}\right)}{50L}}$$

$$0.5\, C_n = \frac{1}{L}\int_0^L \theta(x)\cos\left(\frac{n\pi}{L}\right)x\, dx \qquad n=1,2,3,\ldots$$

$$e^{-50x}\, e^{5L}$$

$$= \frac{\theta_0}{L}\int_0^{0.1L}\cos\left(\frac{n\pi}{L}\right)x\, dx + \frac{\theta_0}{L}\int_{0.1L}^{L} e^{-50\left(x-\frac{L}{10}\right)}\cos\left(\frac{n\pi}{L}\right)x\, dx$$

$$\frac{\theta_0}{L}\int_0^{0.1L}\cos\left(\frac{n\pi}{L}\right)x\, dx = \frac{\theta_0}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{0.1L}$$

$$= \frac{\theta_0}{n\pi}\sin\left(\frac{n\pi}{10}\right)$$

$$\int_{0.1L}^{L} e^{+ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)\Big|_{0.1L}^{L}$$

$$a = -50, \qquad b = \frac{n\pi}{L}$$

$$\Rightarrow \frac{e^{-50x}}{50^2+(n\pi/L)^2}\left(-50\cos\left(\frac{n\pi}{L}\right)x + \frac{n\pi}{L}\sin\frac{n\pi}{L}x\right)\Big|_{0.1L}^{L}$$

$$= \frac{e^{-50L}}{50^2+(n\pi/L)^2}\left((-1)^{n+1}50\right) - \frac{e^{-5L}}{50^2+\left(\frac{n\pi}{L}\right)^2}\left(-50\cos\frac{n\pi}{10} + \frac{n\pi}{L}\sin\frac{n\pi}{10}\right)$$

$$= \frac{e^{-5L}\, 50}{50^2+\left(\frac{n\pi}{L}\right)^2}\left\{(-1)^{n+1}\cancel{50}\, e^{-45L} + \cos\left(\frac{n\pi}{10}\right) - \frac{n\pi}{50L}\sin\frac{n\pi}{10}\right\}$$

$$0.5\, C_n = \frac{\theta_0}{n\pi}\sin\frac{n\pi}{10} + \frac{50(\theta_0/L)}{50^2+\left(\frac{n\pi}{L}\right)^2}\left\{(-1)^{n+1}e^{-45L} + \cos\frac{n\pi}{10} - \frac{n\pi}{50L}\sin\frac{n\pi}{10}\right\}$$

$$0.5\, C_{10} = 0 + \frac{50(\theta_0/L)L^2}{50^2L^2 + 100\pi^2}\left\{-e^{-45L} - 1\right\}$$

$$\Rightarrow \boxed{0.5C_{10} = -\frac{\theta_0 L}{50L^2 + 2\pi^2}\left(1 + e^{-45L}\right)}$$ as required.

SECTION B
4
4a (i):

$$\det(\mathbf{Y}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 4 & a^2 \end{vmatrix} = \begin{vmatrix} 2 & a \\ 4 & a^2 \end{vmatrix} - \begin{vmatrix} 1 & a \\ 1 & a^2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$
$$= (2a^2 - 4a) - (a^2 - a) + 2$$
$$= a^2 - 3a + 2.$$

4a (ii):
$\mathbf{Y}$ is invertible (nonsingular) when $\det(\mathbf{Y}) \neq 0$, which is when $a^2 - 3a + 2 \neq 0 \Rightarrow (a-2)(a-1) \neq 0 \Rightarrow \mathbf{Y}$ is nonsingular when $a \neq 1$ and $a \neq 2$.
4b (i):
The question explicitly asks for a pure **LU** decomposition so there is no need to use pivoting. Therefore, using for instance Doolittle's algorithm, we can directly obtain:

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 10 & 9 & 12 & 1 \\ 4 & 6 & 8 & 3 \\ 8 & 16 & 18 & 16 \end{pmatrix} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix},$$

$$\mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{pmatrix}.$$

4b (ii):
$\det(\mathbf{M}) = \det(\mathbf{LU}) = \det(\mathbf{L})\det(\mathbf{U})$. Since $\mathbf{L}$ and $\mathbf{U}$ are in lower and upper triangular form respectively, $\det(\mathbf{M}) = 2 \cdot 4 \cdot 2 \cdot 9 = 144$.
4c (i):
Form two vectors from the given points, for instance $\mathbf{u} = (4,1,3)^T$ and $\mathbf{v} = (-2,-1,-1)^T$. Since $\mathbf{u}$ and $\mathbf{v}$ are formed from points that lie in the plane, $\mathbf{u}$ and $\mathbf{v}$ will also lie in the plane. Then a normal vector to the plane is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = (2,-2,-2)^T$. The equation of the plane is then $2(x-4) - 2(y-1) - 2(z-3) = 0 \Rightarrow x - y - z = 0$.
4c (ii):
By inspection, a normal vector to the plane is $(1,-1,-1)^T$. Define a line equation $\mathbf{L}(a) = (x+a, y-a, z-a)$ for point (x,y,z). Find a_0 such that $\mathbf{L}(a_0)$ satisfies the

equation of the plane $x - y - z = 0$: $x + a_0 - (y - a_0) - (z - a_0) = 0 \Rightarrow a_0 = \frac{-x+y+z}{3}$. $T(x,y,z) = \left(x + \frac{-x+y+z}{3}, y - \frac{-x+y+z}{3}, z - \frac{-x+y+z}{3}\right)$. T(1,0,0), T(0,1,0) and T(0,0,1) then form rows of the projection matrix:

$$\begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}.$$

4d (i):
Yes, every $\mathbf{X}$ of the form $\begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}$ has an $\mathbf{LU}$ decomposition. Since $\mathbf{X}$ is already upper triangular a trivial $\mathbf{LU}$ decomposition can be obtained by setting $\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{U} = \mathbf{X}$.
4d (ii):
No, the $\mathbf{LU}$ decomposition is not unique. $\mathbf{X} = \mathbf{LU} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} \end{pmatrix} = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}$. We have:

$$\begin{aligned} u_{11} &= 0 \\ u_{12} &= x_1 \\ l_{21}u_{11} &= 0 \\ l_{21}u_{12} + u_{22} &= x_2 \end{aligned}$$

$\therefore u_{11}$ and u_{12} are fixed but l_{21} and $u_{22} = x_2 - l_{21}x_1$ are free.

5

5a:

The points are coplanar. There are many possible ways to show this, for example, from the point set we can find three vectors, for instance, all originating from the first point: $\mathbf{u} = (1,2,-6)^T$, $\mathbf{v} = (2,0,-6)^T$ and $\mathbf{w} = (-2,6,-3)^T$. These three vectors form a parallelepiped and the signed volume of it is the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$. Since the signed volume is zero the points are coplanar.

5b (i):

Yes. $\mathbf{X}$ is singular if it has an eigenvalue of 0. $\mathbf{X}$ has an eigenvalue of 0 iff $\det(\mathbf{X} - 0\mathbf{I}) = \det(\mathbf{X}) = 0$. Since $\mathbf{X}$ is invertible (nonsingular) iff $\det(\mathbf{X}) \neq 0$, $\mathbf{X}$ is invertible iff 0 is not an eigenvalue of $\mathbf{X}$.

5b (ii):

No. The eigenvalues of $\mathbf{X}$ and $\mathbf{X}^T$ are the same. Since $\mathbf{X}$ and $\mathbf{X}^T$ have the same characteristic polynomial and the eigenvalues are roots of the same characteristic polynomial, $\mathbf{X}$ and $\mathbf{X}^T$ have the same eigenvalues:

$$\begin{aligned} \det(\mathbf{X}^T - \lambda \mathbf{I}) &= \det(\mathbf{X}^T - \lambda \mathbf{I}^T) \\ &= \det(\mathbf{X} - \lambda \mathbf{I})^T \\ &= \det(\mathbf{X} - \lambda \mathbf{I}). \end{aligned}$$

5b (iii):

Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ be the row vectors of $\mathbf{X}$. Then $\beta\mathbf{x_1}, \beta\mathbf{x_2}, \ldots, \beta\mathbf{x_n}$ are the row vectors of $\beta\mathbf{X}$. Since scaling a single row by β scales the determinant by β, $\det(\beta\mathbf{X}) = \beta^n \det(\mathbf{X}) = \beta^n \alpha$.

5c:

Using for instance Gram-Schmidt, we obtain:

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

Since $\det(\mathbf{Y}) = -2$, $\mathbf{Y}$ is not singular. Therefore the decomposition is unique.

5d:

By inspection, a normal vector to the plane is $\mathbf{n} = (4,-1,1)^T$. We need two linearly independent vectors in the plane that satisfy $(a,b,c) \cdot (4,-1,1) = 0 \Rightarrow 4a - b + c = 0$, for instance $\mathbf{u} = (0,1,1)^T$ and $\mathbf{v} = (1,4,0)^T$. Using Gram-Schmidt: $\mathbf{b_1} = \frac{\mathbf{u}}{||\mathbf{u}||} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$. Projecting $\mathbf{v}$ onto $\mathbf{b_1}$ we have $\mathbf{p} = \frac{\mathbf{v}\cdot\mathbf{b_1}}{\mathbf{b_1}\cdot\mathbf{b_1}}\mathbf{b_1} = (0,2,2)^T$. By subtracting $\mathbf{p}$ from $\mathbf{v}$ we create a vector perpendicular to $\mathbf{b_1}$. Thus $\mathbf{b_2} = \frac{\mathbf{v}-\mathbf{p}}{||\mathbf{v}-\mathbf{p}||} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)^T$. An orthonormal basis for the plane is therefore $\{\mathbf{b_1}, \mathbf{b_2}\}$. To verify that $\{\mathbf{b_1}, \mathbf{b_2}\}$ is an orthonormal basis let $\mathbf{b_1}$ and $\mathbf{b_2}$ be the columns of a matrix $\mathbf{M}$. Since $\mathbf{M}^T\mathbf{M} = \mathbf{I}$, we have found an orthonormal basis.

6

6a:

$\phi(t)$ for $\alpha = 1$ can be derived as follows:

$$p(T=t) = \frac{\lambda^{\alpha} t^{(\alpha-1)} e^{-\lambda t}}{\Gamma(\alpha)} = \frac{\lambda^{1} t^{(1-1)} e^{-\lambda t}}{\Gamma(1)} = \frac{\lambda e^{-\lambda t}}{1} = \lambda e^{-\lambda t}$$

$$p(T \geq t) = \int_{t}^{\infty} \lambda e^{-\lambda t} dt = e^{-\lambda t}$$

$\phi(t)$ for $\alpha = 1$ is then $\frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$.

Similarly, $\phi(t)$ for $\alpha = 2$ can be derived as follows:

$$p(T=t) = \frac{\lambda^{\alpha} t^{(\alpha-1)} e^{-\lambda t}}{\Gamma(\alpha)} = \frac{\lambda^{2} t^{(2-1)} e^{-\lambda t}}{\Gamma(2)} = \frac{\lambda^{2} t e^{-\lambda t}}{1} = \lambda^{2} t e^{-\lambda t}.$$

$$p(T \geq t) = \int_{t}^{\infty} \lambda^{2} t e^{-\lambda t} dt = e^{-\lambda t}(\lambda t + 1)$$

$\phi(t)$ for $\alpha = 2$ is then $\frac{\lambda^{2} t e^{-\lambda t}}{e^{-\lambda t}(\lambda t+1)} = \frac{\lambda^{2} t}{\lambda t+1}$.

6b:

$\alpha = 1$ results in a memoryless and constant organ failure rate. This is unrealistic since as a biological organism grows older the organ failure rate will increase, which is the case for $\alpha = 2$. Therefore $\alpha = 2$ is more sensible for modelling organ failure rate.

6c:

For $\alpha = 1$ the probability density function is the exponential probability density function with a single rate parameter λ and the organ failure rate $\phi(t) = \lambda$ is memoryless and constant. The organ failure rate for one organ is therefore $\frac{1}{10}$ and for five organs this organ failure rate is then $\frac{5}{10} = \frac{1}{2}$. The survival function is then $p(T \geq t) = e^{-\lambda t} = e^{-\frac{1}{2}2.5} \approx 0.29$. The probability of the organism dying is therefore $1 - p(T \geq t) \approx 0.71$.

Alternatively, the probability $p(T \geq t)$ of one organ failing within the time period is $e^{-2.5\lambda}$. The probability of five organs failing in the time period is $\left(e^{-2.5\lambda}\right)^{5} = e^{-12.5\lambda}$. The probability of the organism dying is therefore $1 - e^{-12.5\lambda}$ (and since an individual organ fails at a rate of $\frac{1}{10}$, $1 - e^{-12.5\lambda} = 1 - e^{-12.5 \cdot 0.1} \approx 0.71$).

6d:

The median survival time will be the solution to $p(T \geq t) = \frac{1}{2}$, which means $\Phi = \frac{1}{2}$. From the data book, $\Phi(0) = \frac{1}{2}$. Thus $z = \frac{\ln(t_{\text{median}}) - \mu}{\sigma} \Rightarrow \ln(t_{\text{median}}) = \mu + z\sigma \Rightarrow t_{\text{median}} = e^{\mu}$.