2019 - IB P7
Vector calculus & Linear Algebra

1) $F = xz^2 i + (x^2y - z^3)j + (2xy + y^2z)k$

(a) $\nabla \cdot F = x^2 + y^2 + z^2$

$\nabla \cdot F = \text{Const} = a^2 \Rightarrow x^2 + y^2 + z^2 = a^2$

This is the equation for a sphere of radius a

$\Rightarrow$ The iso-surfaces are concentric spheres.

(b) $$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= i\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - j\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + k\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

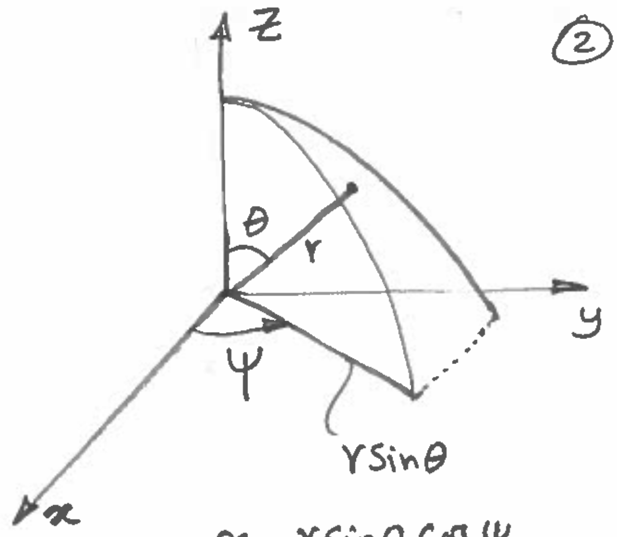
$$= i(2x + 2yz + 3z^2) + j(2xz - 2y) + 2xy\,k$$

$$\boxed{\nabla \times F = (2x + 2yz + 3z^2)i + (2xz - 2y)j + 2xy\,k}$$

(c) Flux $= \oint F \cdot dS$

By Gauss Theorem

$$\oint F \cdot dS = \int \nabla \cdot F \, dV$$



$x = r\sin\theta\cos\psi$

$y = r\sin\theta\sin\psi$

$z = r\cos\theta$

from (a)

$$\nabla \cdot F = x^2 + y^2 + z^2$$
$$= r^2$$

for spherical system

$$dv = r^2 \sin\theta \, d\theta \, d\psi \, dr$$

$0 \le \psi \le 2\pi$

$0 \le \theta \le \pi/2$

for hemisphere

$$\therefore \int \nabla \cdot F \, dV = \int_0^R \int_0^{2\pi} \int_0^{\pi/2} r^2 \, r^2 \sin\theta \, d\theta \, d\psi \, dr$$

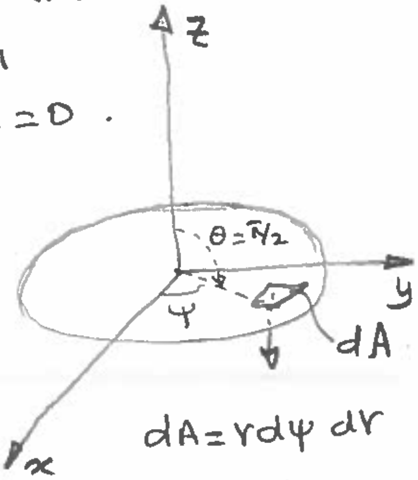
$$= 2\pi \frac{a^5}{5} \int_0^{\pi/2} \sin\theta \, d\theta = 2\pi \frac{a^5}{5} \left. -\cos\theta \right|_0^{\pi/2}$$

$$= \frac{2\pi a^5}{5}$$

$\therefore$ $\boxed{\text{The flux through } S_1 \;\&\; S_2 = \frac{2\pi a^5}{5}}$

(d) Work done is

$$\oint_c F \cdot dl$$ where c is the circle bounding S_1 which is @ $z=0$.



Using Stokes theorem

$$\oint_c F \cdot dl = \int (\nabla \times F) \cdot n \, dA$$

$dA = r\,d\varphi\,dr$

$x = r\cos\varphi$

$y = r\sin\varphi$

$n = -k$

$$\therefore (\nabla \times F) \cdot n = -2xy$$

$$\int (\nabla \times F) \cdot n \, dA = -2 \int_0^a \int_0^{2\pi} r^3 \cos\varphi \sin\varphi \, d\varphi \, dr$$

$$= \frac{a^4}{4} \int_0^{2\pi} \sin 2\varphi \, d\varphi = 0.$$

$$\therefore \boxed{\text{The workdone is } \oint F \cdot dl = 0}$$

(e) From (c) $$\oint F \cdot dS = \int_{S_1} F \cdot dS + \int_{S_2} F \cdot dS$$

From (d) $$\int_{S_1} F \cdot dS = 0,$$ because $F \cdot n = 2xy$

$$\therefore \int_{S_2} F \cdot dS = \oint F \cdot dS = \frac{2\pi a^5}{5}.$$

$$\boxed{\text{The flux through the convex surface is } \frac{2\pi a^5}{5}}$$

comments:
This is a popular question answered quite well by all the candidates. Parts (a) and (b) were answered well. The most common errors found was mostly algebraic for Parts (c) and (d) while evaluating the required integrals. Many students also got the integration limits wrong for spherical and cylindrical polar coordinates required for Parts (c) and (d) respectively. The understanding of concept behind Part (e) was demonstrated well but there were many incorrect final answers as it depends on Parts (c) and (d).

2) $U = (2xy+3)i + (x^2 - 4z)j - 4y\,k$

(a)

(i) The field is conservative if $\nabla \times U = 0$

$$\nabla \times U = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ U_x & U_y & U_z \end{vmatrix}$$

$$= i\left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right) - j\left(\frac{\partial U_z}{\partial x} - \frac{\partial U_x}{\partial z}\right) + k\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)$$

$$= i(-4+4) - j(0-0) + k(2x - 2x) = 0.$$

$\Rightarrow \nabla \times U = 0$, Hence the field is conservative.

(ii) $U = \nabla \phi$

$\Rightarrow \dfrac{\partial \phi}{\partial x} = 2xy + 3 \Rightarrow \phi = x^2y + 3x + f(y,z)$

$\dfrac{\partial \phi}{\partial y} = x^2 - 4z \Rightarrow \phi = x^2y - 4yz + f(x,z)$

$\dfrac{\partial \phi}{\partial z} = -4y \Rightarrow \phi = -4yz + f(y,z)$

$$\Rightarrow \boxed{\phi = x^2y + 3x - 4yz + \text{Const}}$$

This is the scalar potential for U.

(iii) Sine U is conservative $U = \nabla\phi$

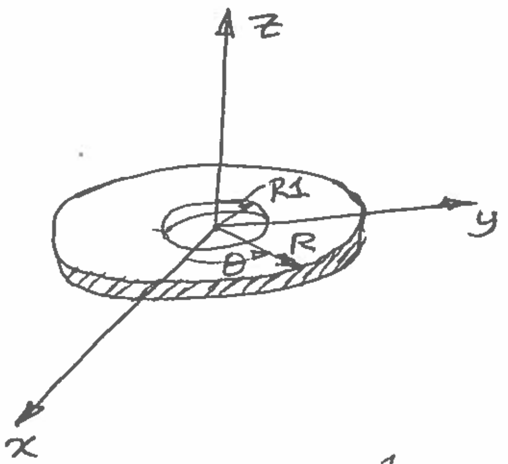
& $\int U \cdot dr$ is independent of path.

$$\Rightarrow \int U \cdot dr = \int \nabla\phi \cdot dr = \int d\phi = \phi \Big|_{(3,-1,2)}^{(2,1,-1)}$$

$$= \phi(2,1,-1) - \phi(3,-1,2) = 6$$

$$\boxed{\int U \cdot dr = 6}$$

(b)



density:

$\rho(x,y,z) = \sqrt{x^2+y^2}$

ρ is independent of z.

use cylindrical polar system

$x = r\cos\theta$; $\quad y = r\sin\theta$ $\qquad R_1 \le r \le R$

$r^2 = x^2+y^2$ $\qquad 0 \le \theta \le 2\pi$

$\qquad 0 \le z \le t$

$dv = r\,d\theta\,dr\,dz$

$\therefore$ (i) Total mass, $M = \int \rho\, dv$

$$= \int_0^t \int_{R_1}^R \int_0^{2\pi} r \cdot r\,d\theta\,dr\,dz = 2\pi t \frac{r^3}{3}\Big|_{R_1}^{R}$$

$$\boxed{M = \frac{2\pi t}{3}\left(R^3 - R_1^3\right)}$$

(ii) $$\int (x^2+y^2)^{3/2}\, dV = \int \sqrt{x^2+y^2}\,(x^2+y^2)\, dV$$

$$= \int \rho\, r^2\, dV$$

$\Rightarrow$ This integral is the moment of the inertia of the plate about the origin (0,0,0)

$$\int_0^t \int_{R_1}^{R} \int_0^{2\pi} r.\, r^2\, r d\theta\, dr\, dz$$

$$= 2\pi t \left. \frac{r^5}{5} \right|_{R_1}^{R} = \frac{2\pi t}{5}(R^5 - R_1^5)$$

$$\therefore \boxed{\int (x^2+y^2)^{3/2}\, dV = \frac{2\pi t}{5}(R^5 - R_1^5)}$$

(iii) K – radius of gyration

$$MK^2 = \int \rho r^2\, dV = \frac{2\pi t}{5}(R^5 - R_1^5)$$

$$\therefore K^2 = \frac{\cancel{2\pi t}}{5} \frac{3}{\cancel{2\pi t}} \frac{(R^5 - R_1^5)}{(R^3 - R_1^3)} = \frac{3}{5} \frac{(R^5 - R_1^5)}{(R^3 - R_1^3)}$$

$$\therefore \boxed{K = \sqrt{\frac{3}{5} \frac{(R^5 - R_1^5)}{(R^3 - R_1^3)}}}$$

comments:

This is the most popular question and the three parts in (a) were answered well as they are straight forward. Part (b) was also answered quite well but algebraic errors while evaluating the integrals in cylindrical polar coordinates were found to be common. The general physical meaning of the integral in Part (b)(ii) was identified by many students but precise explanation was scant.

3) $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$ $y = 0$ at $\begin{cases} x = 0 \\ x = L \end{cases}$

$\dot{y}(0) = 0$, ie $\dot{y}(t=0) = 0$.

(a)

$$y = G(t)\, F(x)$$

$$\Rightarrow \frac{1}{c^2}\frac{\ddot{G}}{G} = \frac{F''}{F} = -\lambda^2$$

$$F'' + \lambda^2 F = 0; \quad F(0) = F(L) = 0$$

$$\Rightarrow F(x) = A\cos\lambda x + B\sin\lambda x$$

$F(0) = 0 \Rightarrow A = 0;$ $F(L) = 0 \Rightarrow$ $\sin\lambda L = 0$ $\Rightarrow \boxed{\lambda = \frac{n\pi}{L}}$

$$\therefore F(x) = B\sin\left(\frac{n\pi}{L}x\right) \quad n = 0, 1, 2, \ldots \infty$$

$$\ddot{G} + (\lambda c)^2 G = 0 \qquad \text{Let } \omega_n = \frac{n\pi c}{L}$$

$$\Rightarrow \ddot{G} + \omega_n^2 G = 0 \Rightarrow G(t) = D\cos\omega_n t + E\sin\omega_n t.$$

$$\dot{y}(t=0) = 0 \Rightarrow \dot{G}(0) = 0$$

$$\Rightarrow \dot{G} = -D\,\omega_n \sin\omega_n t + E\,\omega_n\cos\omega_n t$$

$$\dot{G}(0) \Rightarrow E = 0$$

$$\therefore G(t) = D\cos\omega_n t.$$

$$\therefore y(t,x) = \sum_{n=0}^{\infty} A_n \cos\omega_n t \sin\left(\frac{n\pi x}{L}\right)$$

$n = 0$ gives $y(t,x) = 0$; $\therefore$ $\boxed{y(t,x) = \sum_{n=1}^{\infty} B_n \cos\omega_n t \sin\left(\frac{n\pi x}{L}\right)}$

as required.

(b) $\omega_n = \dfrac{n\pi c}{L}$

(c) $y(t,x) = \sum_{n=1}^{\infty} A_n \cos\omega_n t \, \sin\left(\dfrac{n\pi x}{L}\right)$

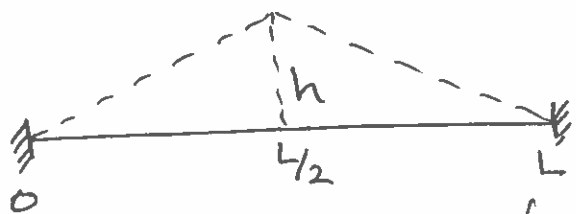
use the initial condition $y(0,x) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

This is half - Sine Series

Thus $\boxed{A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\dfrac{n\pi x}{L}\right) dx.}$

(d)



@ $t = 0$.

$$\therefore f(x) = y(0,x) = \begin{cases} \dfrac{2h}{L}x & \text{for } 0 \le x \le L/2 \\ \dfrac{2h}{L}(L-x) & L/2 \le x \le L \end{cases}$$

$$\therefore A_n = \frac{2}{L}\int_0^{L/2} \frac{2h}{L} x \sin\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L}\int_{L/2}^{L} \frac{2h}{L}(L-x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{4h}{L^2}\left\{ \int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx + L\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) dx - \int_{L/2}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx \right\}$$

By parts:

$$\int x \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{L/2} + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^{L/2}$$

$$= -\frac{L^2}{n\pi}\left(\cos\frac{n\pi}{2} - 0\right) + \frac{L^2}{n^2\pi^2}\left\{\sin\frac{n\pi}{2} - 0\right\}$$

$$= +\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{L/2}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L} + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L}$$

$$= -\frac{L^2}{n\pi}\left(\cos n\pi - \frac{1}{2}\cancelto{0}{\cos\frac{n\pi}{2}}\right) + \frac{L^2}{n^2\pi^2}\left\{\cancelto{0}{\sin n\pi} - \sin\frac{n\pi}{2}\right\}$$

$$= -\frac{L^2}{n\pi}\cos n\pi - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L}$$

$$= -\frac{L}{n\pi}\left\{\cos n\pi - \cancelto{0}{\cos\frac{n\pi}{2}}\right\}$$

$$= -\frac{L}{n\pi}\cos n\pi$$

$$\therefore A_n = \frac{4h}{L^2}\left\{ \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{L^2}{n\pi}\cos n\pi} + \cancel{\frac{L^2}{n\pi}\cos n\pi} + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}$$

$$\boxed{A_n = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)}$$

$$\therefore A_1 = \frac{8h}{\pi^2}\;; \qquad A_3 = \frac{-8h}{9\pi^2}$$

As required.

comments:

This is the least popular question. Parts (b) and (c) are answered well. The mechanistic aspects required for Part (a) were answered mostly well but the reason to exclude n = 0 mode was explained only by about 20% of the students attempted this question. Part (d) was not answered well in general and it required evaluation of few integrals – algebraic errors were common and it is likely that most students ran out of time.

4 (a) $\det(A) = k^2 + \cancel{1} - \cancel{1} - 2 = k^2 - 2$

to have solutions $\det(A) \neq 0$

$k^2 - 2 \neq 0 \quad k \neq \mp\sqrt{2}$

for $k \neq \mp\sqrt{2}$ solve by elimination:

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & k^2+1 \end{pmatrix} \underline{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & k^2-1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & k^2-2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

and back substitution:

$$x_3 = -\frac{2}{k^2-2}$$

$$x_2 + x_3 = 1 \implies x_2 = 1 - x_3 = 1 + \frac{2}{k^2-2} = \frac{k^2}{k^2-2}$$

$$x_1 - x_2 + 2x_3 = 0 \implies x_1 = x_2 - 2x_3 = \frac{k^2+4}{k^2-2}$$

$$\underline{x} = \begin{pmatrix} \frac{k^2+4}{k^2-2} \\ \frac{k^2}{k^2-2} \\ \frac{-2}{k^2-2} \end{pmatrix}$$

4 (b)

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{array}{c|c} & \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \\ \hline \begin{matrix} 1 \\ -1 \\ 0 \end{matrix} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A \quad = \quad \ell_1 u_1 \quad + \quad R_1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & -1 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ -1 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} + \begin{array}{c|c} & \begin{matrix} 0 & 0 & 1 \end{matrix} \\ \hline \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$R_1 \quad = \quad \ell_2 u_2 \quad + \quad R_2 \left(= \ell_3 u_3\right)$$

$$A = \ell_1 u_1 + \ell_2 u_2 + \ell_3 u_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\qquad\quad L \qquad\qquad\qquad\quad U$$

$$Ax = b \implies LUx = b \implies Lc = b \quad \text{where } c = Ux$$

for $b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad x_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad c_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

4 (b) continued

let

$$X = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A \cdot X = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

since $AX = I$ $\quad X = A^{-1}$

4 (c)

if $a \neq 0 \Rightarrow L = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} a & b \\ 0 & d - bc/a \end{bmatrix}$ ✓

if $a = 0$ and $c = 0 \Rightarrow L = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$ ✓

with arbitrary e

if $a = 0$ and $c \neq 0$ it is impossible to factor A with non zero diagonals in L and U.

so the answer is <u>no</u>

4 (d) find eigenvalues of A ~~from~~

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 6-\lambda \end{pmatrix} = 0$$

4 (d) continued

$$(1-\lambda)(1-\lambda)(6-\lambda)-6(1-\lambda)=0$$

$$(1-\lambda)\left[(1-\lambda)(6-\lambda)-6\right]=$$

$$=(1-\lambda)\left[\cancel{6}-\lambda-6\lambda+\lambda^2-\cancel{6}\right]=$$

$$=\lambda(1-\lambda)(\lambda-7)=0 \implies \lambda_1=0,\quad \lambda_2=1 \quad \lambda_3=7$$

the corresponding eigen vectors are found from $(A-\lambda_i I)\underline{v}_i = 0$

$$\lambda=0 \qquad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 6 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \qquad v_0 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda=1 \qquad \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda=7 \qquad \begin{pmatrix} -6 & 0 & 2 \\ 0 & -6 & 0 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v_7 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

matrix P is:

$$P = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ v_0 & v_1 & v_7 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

This is invertible because $\det(P) = -7 \neq 0$

4 (d) continued

and $P^{-1} = \frac{1}{\det(P)} C^T$

where $C = \begin{pmatrix} 3 & 0 & -1 \\ 0 & -7 & 0 \\ -1 & 0 & -2 \end{pmatrix} = C^T$

$$P^{-1} = \begin{pmatrix} -\frac{3}{7} & 0 & \frac{1}{7} \\ 0 & 1 & 0 \\ \frac{1}{7} & 0 & \frac{2}{7} \end{pmatrix}$$

it is easy to verify that

$$PP^{-1} = I$$

and that

$$D = P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

which has the eigenvalues on the main diagonal

comments:
A surprisingly large number of candidates failed to state correctly the condition for the system of linear equations to have solutions. Most candidates could solve the more straightforward part on LU factorisation, whereas less candidates seemed comfortable with the more theoretical one. The part on the eigenvalue problem was generally well tackled except for some minor algebraic mistakes.

5 (a)

If x is eigenvector of A:

$Ax = \lambda x$

$$Ax_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

so x_1 is eigenvector of A with eigenvalue 4

$$Ax_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

x_2 is not eigenvector of A because it is not possible to find $\lambda \in \mathbb{R}$! $Ax_2 = \lambda x_2$

$$Ax_3 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

x_3 is eigenvector of A with eigenvalue 3

5 (b) The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -2$ (A is upper triangular and its eigenvalues are the values on the diagonal)

The corresponding eigen vectors are

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_{-2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

and A can be diagonalised as:

$$D = P^{-1}AP \quad \text{with } D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\text{and } P = \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}$$

5 (b) continued

Therefore A can be written as:

$$\boxed{A = P D P^{-1}}$$

and $A^{100} = P D^{100} P^{-1}$

or $A^{100} = \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix} 2^{100} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{100} \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}^{-1} =$

$$= 2^{100} \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}^{-1} =$$

$$= 2^{100} \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}^{-1} =$$

$$= 2^{100} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2^{100} I$$

5 (c) compute square matrix:

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

and $A^T b$:

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

solve $A^T A \hat{v} = A^T b$ to find $\hat{v}$

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

5 (c) continued

which yields

$$\hat{v} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

The projection of b onto the column space of A is

$$p = A\hat{v}$$

$$p = \begin{pmatrix} 1 & 0 \\ 1 & +1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

The projection matrix is

$$P = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\det(A^TA) = 15 - 9 = 6$$

$$(A^TA)^{-1} = \frac{1}{6}\begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix}$$

$$P = \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$\frac{1}{6}\begin{pmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

5 (d) if 1 is eigen value of R then

$$\det(R - 1I) = 0$$

$$\det(R - I) = 0$$

from $(R-I)R^T = RR^T - R^T = I - R^T$

take determinant

$$\det(R-I)\underbrace{\det R}_{+1} = \det(I - R^T) =$$

$$= \det\left((I-R)^T\right) = \det(I-R) =$$

$$= \det(-1(R-I))$$

we have shown that

$$\det(R-1) = -\det(R-I) \Rightarrow \det(R-I)=0$$

* This is actually Euler's Theorem: every rotation can be represented as a rotation about one axis, e

$$Re = 1e$$

comments:

The first two parts were answered generally well as they were very straightforward (eigenvalues and eigenvectors of an L matrix, diagonalisation and power of a 2x2 matrix). Very few candidates managed to find the best fit solution and projection onto the column space required for part (c). The part (d) required to demonstrate Euler's theorem that every rotation can be represented as a rotation about one axis without using the characteristic polynomial. Although almost all candidates stated that they had obtained the required result, hardly any of them actually managed to produce a working demonstration.

6 (a) it must be

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 \qquad \text{or:}$$

$$k \int_0^{\infty} x\, e^{-\frac{x^2}{2}}\,dx = 1$$

$$\int_0^{\infty} x\, e^{-\frac{x^2}{2}}\,dx = -\int_0^{\infty} \frac{d}{dx} e^{-\frac{x^2}{2}}\,dx = -e^{-\frac{x^2}{2}}\Big|_0^{+\infty} = 1$$

it follows $k \cdot 1 = 1 \implies k = 1$

6 (b) if the train arrives between 8:33 and 8:35, the delay in minutes is

$3 < X \leq 5$

$$P(3 < X \leq 5) = \int_3^5 x\, e^{-\frac{x^2}{2}}\,dx = -e^{-\frac{x^2}{2}}\Big|_3^5 = e^{-9/2} - e^{-25/2} =$$

$$1.11 \times 10^{-2} = 1.11\,\%$$

average delay E:

6 (c) $$E(X) = \int_0^{+\infty} x^2 e^{-\frac{x^2}{2}}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2}} =$$

$$\frac{\sqrt{2\pi}}{2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2}} = \frac{\sqrt{2\pi}}{2} = 1.25 \text{ min}$$

so the average arrival time is:

8:31' 15''

6 (d) Let Y be the number of times the train is more than 5 min late in 10 days

If $p = P(X>5)$ then Y is binomial with $n = 10$ and $p = P(X>5)$:

It follows that:

$$P(Y \leq 2) = \sum_{k=0}^{2} \binom{10}{k} p^k (1-p)^{10-k}$$

In this case:

$$P(X>5) = \int_5^\infty x e^{-\frac{x^2}{2}} dx = -e^{-\frac{x^2}{2}} \Big|_5^\infty = e^{-\frac{25}{2}}$$

and the probability that M will not be blamed is:

$$P(Y \leq 2) = \left(1-e^{-\frac{25}{2}}\right)^{10} + 10 e^{-\frac{25}{2}}\left(1-e^{-\frac{25}{2}}\right)^9 + 45 e^{-25}\left(1-e^{-\frac{25}{2}}\right)^8$$

4

comments:
This question required verifying that an assigned probability density function satisfies the properties of a probability density function, and working out a number of characteristics of the distribution. The candidates were generally comfortable with the required integration by parts, and were not daunted by the numerical value of the train being more than 5 min late being exceedingly small. Typically, very well answered, probably too easy.