① (a)

(i) $\underline{F} = -y\underline{i} + x\underline{j} + 5\underline{k}$ $\quad \nabla\cdot\underline{F} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} + \frac{\partial(5)}{\partial z} = 0$ [1]

$\nabla\cdot F = 0$ everywhere hence $\underline{F}$ is Solenoidal

$$\Rightarrow \nabla\times\underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 5 \end{vmatrix} =$$

$$i\left(\frac{\partial 5}{\partial y} - \frac{\partial x}{\partial z}\right) - j\left(\frac{\partial 5}{\partial x} - \frac{\partial y}{\partial z}\right) + k\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) = \underline{2k}$$ [1]

$\Rightarrow \iint_{S_1} (\nabla\times\underline{F})\cdot d\underline{A} = \iint 2k\cdot d\underline{A} \quad \{d\underline{A} = -\underline{k}\,dA\} = \underline{-2\pi a^2}$ [1]

(ii) Since $dA = -\underline{k}\,dA$ and $F_z = 5$, $\iint_{S_1} \underline{F}\cdot d\underline{A} = \underline{-5\pi a^2}$ [1]

Solenoidal $\therefore \oiint_S \underline{F}\cdot d\underline{A} = 0$

$\therefore \iint_{S_1} \underline{F}\cdot d\underline{A} + \iint_{S_2} F\cdot dA = 0; \ \therefore \iint_{S_2} \underline{F}\cdot d\underline{A} = \underline{5\pi a^2}$ [2]

(iii)

$\underline{i} = \underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta, \quad \underline{j} = \underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta$

$x = r\cos\theta,\ y = r\sin\theta \rightarrow \underline{F} = -r\sin\theta(\underline{e}_r\cos\theta - \underline{e}_\theta\sin\theta) + r\cos\theta(\underline{e}_r\sin\theta + \underline{e}_\theta\cos\theta) + 5\underline{e}_z$

$\Rightarrow F = \underline{e}_r(-r\sin\theta\cos\theta + r\sin\theta\cos\theta) + r\underline{e}_\theta\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1} + 5\underline{e}_z = \underline{r\underline{e}_\theta + 5\underline{e}_z}$ [2]

$\Rightarrow$ We have solid body rotation around $\underline{e}_z$ with a constant component in the $\underline{e}_z$ direction i.e. helical

[2] $\Rightarrow \int_c \underline{F}\cdot d\underline{l} = a\cdot 2\pi a = \underline{2\pi a^2}$ [2]

(iv) Stokes

$\iint_{S_2} (\nabla\times\underline{F})\cdot d\underline{A} = \int_c \underline{F}\cdot d\underline{l} = \underline{2\pi a^2}$ ← Stokes [2]

Gauss

$\iint_{S_1} (\nabla\times\underline{F})\cdot d\underline{A}$ from Part (i) $= -2\pi a^2$

Note $\oiint (\nabla\times\underline{F})\, d\underline{A} = 0$ Since curl field Solenoidal $\therefore \iint_{S_1} (\nabla\times\underline{F})\cdot dA + \iint_{S_2} (\nabla\times\underline{F})\cdot dA = 0$

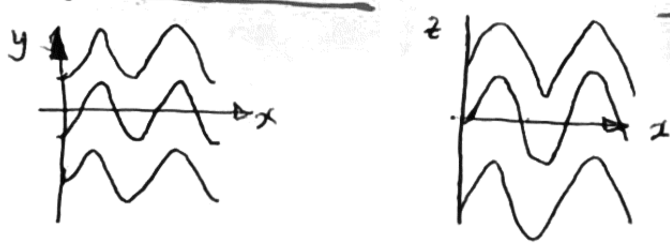
$\therefore \iint_{S_2} (\nabla\times\underline{F})\, d\underline{A} = \underline{2\pi a^2}$ [2]

1b) $\underline{G} = \underline{i} + \sin x \, \underline{j} + \cos x \, \underline{k}$, $\quad \dfrac{dx}{V_x} = \dfrac{dy}{V_y} = \dfrac{dz}{V_z}$

∴ For field lines

$$\frac{dy}{dx} = \sin x \qquad \frac{dz}{dx} = \cos x$$

$$\underline{y = -\cos x + C_1} \qquad \underline{z = \sin x + C_2}$$



[3]

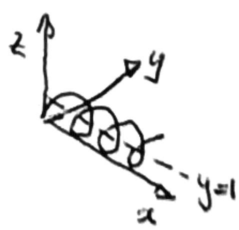
Field line through $(0, 0, 0)$

$$0 = -1 + C_1 \quad \therefore \quad C_1 = 1$$

$$0 = 0 + C_2 \quad \therefore \quad C_2 = 0$$

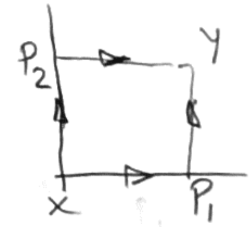
$$\underline{y = -\cos x + 1}, \quad \underline{z = \sin x}$$

[3]



[3]

2) i) $I_1 = \int_0^1 3y^2\,dy = [y^3]_0^1 = 1$



$I_2 = \int_0^1 2x\,dx = [x^2]_0^1 = 1$

[3]

ii) $\underline{\nabla \times A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3 & 3x^2y^2 & 0 \end{vmatrix}$

$i(0) + j\,0 + k\left(\frac{\partial}{\partial x}3x^2y^2 - \frac{\partial}{\partial y}2xy^3\right)$

$y^2\,6x - 2x\,3y^2 = 0$

[2]

So $\nabla \times A = 0 \Rightarrow$ there exists ϕ such that $\underline{A} = -\nabla\phi$

$= -\left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j\right)$, $-\nabla\phi = 2xy^3, 3x^2y^2$

$-\frac{\partial\phi}{\partial x} = 2xy^3 \qquad \int \partial\phi = -\int 2xy^3\,\partial x = x^2y^3 + c_1$

$-\frac{\partial\phi}{\partial y} = 3x^2y^2 \qquad \int \partial\phi = -\int 3x^2y^2\,\partial y = x^2y^3 + c_2$

$\underline{\phi = x^2y^3}$

[2]

iii) $u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \qquad \frac{v}{\partial y} = \frac{u}{\partial x}, \qquad \frac{3x^2y^2}{\partial y} = \frac{2xy^3}{\partial x}$

$\int 3x$ [illegible] $\partial x = \int 2$ [illegible] $y\,\partial y$ + const

(2) Continued

$$\frac{3x^2}{\cancel{2}} = \frac{2y^2}{\cancel{2}} + C_2$$

$$3x^2 - 2y^2 = \text{constant}$$ [3]

(3) continued

$T = X(x)\,Y(y)$

$$Y \frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{d^2X}{X\,dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dy^2} = K$$

K is constant which can be equal to, < or > zero

Try $K < 0$, suppose $K = -\alpha^2$

$$T = (A\cos\alpha x + B\sin\alpha x)(C\cosh\alpha y + D\sinh\alpha y)$$

$T(0,y) = 0 \rightarrow A(C\cosh\alpha y + D\sinh\alpha y) = 0$

$C = D = 0 \rightarrow T = 0$

or $A = 0 \rightarrow T = B\sin\alpha x\,(C\cosh\alpha y + D\sinh\alpha y)$

$T(x,0) = 0 \rightarrow BC\sin\alpha x = 0$

$B = 0 \rightarrow T = 0$

continue with $C = 0$

$$\underline{T = BD\sin\alpha x\ \sinh\alpha y}$$

$T(a,y) = 0 \rightarrow BD\sin\alpha a\ \sinh\alpha y$

$B = 0$ or $D = 0 \rightarrow T = 0$

$\sin\alpha a = 0 \quad \alpha = \frac{n\pi}{a}$

$$T_n(x,y) = BD\sin\frac{n\pi}{a}\ \sinh\frac{n\pi}{a}y$$

② Continued

- From the 1st 3 boundary conditions we have

$$T(x,y) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

The last boundary condition is

$$T(x,b) = T_0 \sin\frac{\pi x}{a}$$

giving

$$T_0 \frac{\sin \pi x}{a} = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

This shows $n=1$ only so

$$k_1 = \frac{T_0}{\sinh\frac{\pi b}{a}}$$

Hence, final solution is

$$T(x,y) = \frac{T_0}{\sinh\frac{\pi b}{a}} \sin\frac{\pi x}{a} \sinh\frac{\pi y}{a} \quad [15]$$

(3) a) The convective flux into the volume is $-\int \rho\phi \underline{u} \cdot d\underline{s}$
the negative sign accounts for S being an outward facing normal. Similarly for diffusion we have
$-\int (-\gamma \nabla\phi)\, d\underline{s}$
Again the minus sign accounts for the outward facing normal (the inner one is because ϕ is diffused in the direction $-\nabla\phi$)
These two fluxes must sum to give the rate of accumulation / loss of ϕ in the volume V giving

$$\frac{\partial}{\partial t}\int_V (\rho\phi)\, dv = \oint_S \rho\phi u \cdot ds - \oint_S (-\gamma \nabla\phi)\, ds$$ [5]

b) From Gauss's theorem

$$\int_S [\rho\phi u] \cdot ds = \int_V \nabla \cdot [\rho\phi u]\, dv$$

and hence

$$\int_S [-\gamma \nabla\phi] \cdot ds = \int \gamma \nabla \cdot (\nabla\phi)\, dv = \int \gamma \nabla^2 \phi\, dV$$

Therefore, the surface integral equation becomes the following volume integral

$$\frac{\partial}{\partial t}\int_V (\rho\phi)\, dV = -\int_V \nabla \cdot (\rho\phi u)\, dv - \int_V -\gamma \nabla^2 \phi\, dV$$ [3]

c) Integral holds for every volume, hence

$$\frac{\partial \rho\phi}{\partial t} + \nabla \cdot \rho u \phi = \gamma \nabla^2 \phi$$

For constant density

$$\rho \frac{\partial \phi}{\partial t} + \rho \nabla \cdot u\phi = \gamma \nabla^2 \phi$$

③ continued

or: $\rho \frac{\partial \phi}{\partial t} + \rho (u \cdot \nabla \phi + \phi \nabla u) = \gamma \nabla^2 \phi$

but $\nabla u = 0$, for incompressible fluid. Hence,

$$\rho \frac{\partial \phi}{\partial t} + \rho u \cdot \nabla \phi = \gamma \nabla^2 \phi$$

[7]

d) If $\underline{u} = 0$ above becomes

$$\frac{\partial \phi}{\partial t} = \frac{\gamma}{\rho} \nabla^2 \phi$$

$$\eta = x (2 \gamma t / \rho)^{-1/2}$$

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x} \frac{d}{d\eta} = \left(2 \gamma \frac{t}{\rho} \right)^{-1/2} \frac{d}{d\eta}$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{2 \gamma t}{\rho} \right)^{-1} \frac{d^2}{d\eta^2}$$

$$\frac{\partial}{\partial t} = \frac{-x}{2} \left(\frac{2\gamma}{\rho} \right)^{-1/2} t^{-3/2} \frac{d}{d\eta} = \frac{-\eta}{2t} \frac{d}{d\eta}$$

$$\left. \begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{-\eta}{2t} \frac{d\phi}{d\eta} \\ \frac{\gamma}{\rho} \frac{\partial^2 \phi}{\partial x^2} &= \frac{\cancel{\gamma}}{\cancel{\rho}} \frac{\cancel{\rho}}{2 \cancel{\gamma} t} \frac{d^2 \phi}{d\eta^2} = \frac{1}{2t} \frac{d^2 \phi}{d\eta^2} \end{aligned} \right\} \frac{d^2 \phi}{d\eta^2} + \eta \frac{d\phi}{d\eta} = 0$$

[10]

1.a

$$A = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

The characteristic polynomial of A is:

$$\det |A - \lambda I| = \det \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2$$

and therefore the eigenvalues of A are:

$$\lambda_1 = 1 \qquad \lambda_{2,3} = 2$$

Note: could have found λ_1 and $\lambda_{2,3}$ directly as the elements on the diagonal, as A is (lower) triangular.

The corresponding eigenvectors are found from:

$$(A - \lambda_i I)\underline{v} = \underline{0}$$

for $\lambda_1 = 1$ this yields

$$\begin{vmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \quad \text{or} \quad \begin{aligned} -x + y &= 0 \\ -x + z &= 0 \end{aligned}$$

one solution is $x = 1 \quad y = 1 \quad z = 1$ or $\underline{v}_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

for $\lambda_{2,3} = 2$

$$\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \quad \text{or} \quad x = 0$$

with solutions

$x = 0 \quad y = 1 \quad z = 0$

$x = 0 \quad y = 0 \quad z = 1$

that is $\underline{v}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad \underline{v}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

Can choose P as the matrix that has the eigenvalues as columns:

$$P = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$, invertible as $\det P = 1 \neq 0$

$$P^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \qquad B = P^{-1}AP = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

1.b First write the five equations $A\underline{v} = \underline{y}$ in the three unknowns $\underline{v} = \begin{pmatrix} C \\ D \\ E \end{pmatrix}$ for a parabola to go through the five points. No solution because no such parabola exists. Solve $A^T A \hat{\underline{v}} = A^T \underline{y}$

$$C + D(-2) + E(-2)^2 = 0$$
$$C + D(-1) + E(-1)^2 = 0$$
$$C + D(0) + E(0)^2 = 1$$
$$C + D(1) + E(1)^2 = 0$$
$$C + D(2) + E(2)^2 = 0$$

$$A = \begin{vmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$A^T A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{vmatrix} \begin{vmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{vmatrix}$$

N.B.: The zeroes in $A^T A$ mean that column 2 of A is orthogonal to columns 1 and 3

The best fit C, D, E come from

$A^T A \hat{v} = A^T \underline{y}$ and D is uncoupled

$$\begin{vmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{vmatrix} \begin{vmatrix} C \\ D \\ E \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

leading to:

$$C = \frac{34}{70}$$

$$D = 0$$

$$E = \frac{-1}{7}$$

1.c Let $\underline{v}(\neq \underline{0})$ be an eigenvector of A with eigenvalue λ, i.e.: (3)

$$A\underline{v} = \lambda \underline{v}$$

It follows that

$$(A-2I)v = A\underline{v} - 2\underline{v} = \lambda \underline{v} - 2\underline{v} = (\lambda - 2)\underline{v}$$

This means that $\underline{v}$ is also an eigenvector of $(A-2I)$ with eigenvalue $(\lambda-2)$

Similarly an eigenvector of A with eigenvalue μ is also an eigenvector of $A-2I$ with eigenvalue $\mu - 2$

Because $A-2I$ can only have 2 eigenvalues these are $\lambda-2$ and $\mu-2$

1.d

ii.d.

$$A = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

we need to find an orthonormal basis for the column space of A. The columns of A are:

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

using Gram-Schmidt

$$u_1 = a_1 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \qquad e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \qquad e_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$u_2 = a_2 - (a_2 \cdot e_1)\, e_1 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$e_2 = \frac{u_2}{\|u_2\|} \qquad \|u_2\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$$

$$e_2 = \frac{2}{\sqrt{6}} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

$$u_3 = a_3 - (a_3 \cdot e_1)\, e_1 - (a_3 \cdot e_2) \cdot e_2 =$$

$$= a_3 - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - \frac{1}{\sqrt{6}} \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/6 \\ -1/6 \\ 2/6 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 - 1/2 - 1/6 \\ 1 - 1/2 + 1/6 \\ 1 - 2/6 \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{pmatrix} \frac{-4}{6} \\ \frac{6-3+1}{6} \\ \frac{6-2}{6} \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$\|u_3\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}} = \sqrt{\frac{12}{9}} = \sqrt{\frac{2^2\cdot 3}{3^2}} = \frac{2}{3}\sqrt{3}$$

$$12 \left(= \begin{array}{r|l} 12 & 2 \\ 6 & 2 \\ 3 & 3 \\ 1 & \end{array} = 2^2\cdot 3\right.$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{3}{2\sqrt{3}} \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$e_1 = \begin{vmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{vmatrix} \qquad e_2 = \begin{vmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{vmatrix} \qquad e_3 = \begin{vmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{vmatrix}$$

$$Q = \begin{vmatrix} \uparrow & \uparrow & \uparrow \\ e_1 & e_2 & e_3 \\ \downarrow & \downarrow & \downarrow \end{vmatrix} = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{vmatrix}$$

$$R = \begin{vmatrix} a_1\cdot e_1 & a_2\cdot e_1 & a_3\cdot e_1 \\ 0 & a_2\cdot e_2 & a_3\cdot e_2 \\ 0 & 0 & a_3\cdot e_3 \end{vmatrix} = \begin{vmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{vmatrix}$$

check

$$QR = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{vmatrix} \begin{vmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & \frac{1}{2}+\frac{3}{6} & \frac{1}{2}+\frac{1}{6}-\frac{2}{3} \\ 1 & \frac{1}{2}-\frac{3}{6} & \frac{1}{2}-\frac{1}{6}+\frac{2}{3} \\ 0 & 1 & \frac{2}{6}+\frac{2}{3} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \checkmark$$

2.a.

$$A = \begin{vmatrix} k & -1 & k-1 \\ 1 & 1 & 1-k \\ 2 & 1 & 6 \end{vmatrix} \quad b = \begin{vmatrix} k \\ k+2 \\ 1 \end{vmatrix}$$

solve $A\underline{x} = \underline{b}$

$$\det A = k\left[6-(1-k)\right] + \left[6-2(1-k)\right] + (k-1)\left[1-2\right] =$$

$$= k\left[5+k\right] + \left[4+2k\right] - \left[k-1\right] =$$

$$= 5k + k^2 + 4 + 2k - k + 1 =$$

$$= k^2 + 6k + 5$$

$\det A = 0$ $\quad \Delta = 36 - 20 = 16$

$$k_{12} = \frac{-6 \mp \sqrt{16}}{2} \quad \begin{cases} k_1 = -5 \\ k_2 = -1 \end{cases}$$

for $k \neq -5$ and $k \neq -1$, the solution of $A\underline{x} = \underline{b}$ is unique and can be found e.g. by Gaussian elimination

$$\begin{vmatrix} k & -1 & k-1 \\ 0 & 1 + \frac{1}{k} & (1-k) - \frac{(k-1)}{k} \\ 0 & 1 + \frac{2}{k} & 6 - \frac{2(k-1)}{k} \end{vmatrix} \quad \begin{vmatrix} k \\ (k+2) - 1 \\ 1 - 2 \end{vmatrix}$$

$$\begin{vmatrix} k & -1 & k-1 \\ 0 & \frac{k+1}{k} & \frac{k-k^2-k+1}{k} \\ 0 & \frac{k+2}{k} & \frac{6k-2k+2}{k} \end{vmatrix} \quad \begin{vmatrix} k \\ k+1 \\ -1 \end{vmatrix}$$

$$\left(\begin{array}{ccc|c} k & -1 & k-1 & k \\ 0 & \frac{k+1}{k} & \frac{1-k^2}{k} & k+1 \\ 0 & \frac{k+2}{k} & \frac{4k+2}{k} & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} k & -1 & k-1 & k \\ 0 & \frac{k+1}{k} & \frac{1-k^2}{k} & k+1 \\ 0 & 0 & \frac{4k+2}{k} - \frac{k+2}{\cancel{k+1}} \overbrace{\frac{1-k^2}{k}}^{(1-k)\cancel{(1+k)}} & -1-\frac{(k+2)}{\cancel{k+1}}\cancel{(k+1)} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} k & -1 & k-1 & k \\ 0 & \frac{k+1}{k} & \frac{1-k^2}{k} & k+1 \\ 0 & 0 & \frac{4k+2}{k} - \frac{(k+2)}{k}(1-k) & -(3+k) \end{array}\right)$$

$$\left(\begin{array}{ccc|c} k & -1 & k-1 & k \\ 0 & \frac{k+1}{k} & \frac{1-k^2}{k} & k+1 \\ 0 & 0 & \frac{4k+2-\overbrace{(k+2-k^2-2k)}^{-k^2+2-k}}{k} & -(3+k) \end{array}\right)$$

$4k+\cancel{2}+k^2-\cancel{2}+k$

$5k+k^2 \quad k(5+k)$

$$\left(\begin{array}{ccc|c} k & -1 & k-1 & k \\ 0 & \frac{k+1}{k} & \frac{1-k^2}{k} & k+1 \\ 0 & 0 & 5+k & -(3+k) \end{array}\right)$$

$$(5+k)\,z = -(3+k)$$

$$z = -\frac{3+k}{5+k} \qquad k \neq -5$$

$$\frac{k+1}{k}\,y - \frac{1-k^2}{k}\cdot\frac{3+k}{5+k} = k+1$$

$$\frac{k+1}{k}\,y = (k+1) + \frac{(1-k)(k+1)(3+k)}{k(5+k)}$$

$$\frac{\cancel{k+1}}{\cancel{k}}\,y = \frac{k(\cancel{k+1})(5+k) + (3+k)(1-k)(\cancel{k+1})}{\cancel{k}(5+k)}$$

$$y = \frac{k(5+k) + (3+k)(1-k)}{(5+k)} =$$

$$= 5k + \cancel{k^2} + 3 + k - 3k - \cancel{k^2} =$$

$$= \frac{3k+3}{5+k} = \frac{3(k+1)}{(5+k)}$$

$$kx - y + (k-1)\,z = k$$

$$kx - \frac{3(k+1)}{5+k} - (k-1)\frac{(3+k)}{(5+k)} = k$$

$$kx = k + \frac{3(k+1)}{5+k} + \frac{(k-1)(3+k)}{5+k}$$

$$kx = \frac{5k + k^2 + 3k + \cancel{3} + 3k + k^2 - \cancel{3} - k}{5+k}$$

$$kx = \frac{2k^2+10k}{5+k} \Rightarrow x = \frac{2\cancel{k}(\cancel{k+5})}{\cancel{k}(\cancel{k+5})} = 2$$

$$\begin{cases} x = 2 \\ y = \dfrac{3(k+1)}{5+k} \\ z = -\dfrac{3+k}{5+k} \end{cases}$$

for $k=-5$ there are no solutions

for $k=-1$ we have

$$A = \begin{vmatrix} -1 & -1 & -2 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{vmatrix} \quad \underline{b} = \begin{vmatrix} -1 \\ 1 \\ 1 \end{vmatrix}$$

it is clear that the second equation is equal to the first equation by -1

The system has an infinity of solutions that can be found by Gaussian elimination

$$\begin{vmatrix} -1 & -1 & -2 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & -1 & 2 \end{vmatrix} \begin{vmatrix} -1 \\ \cancel{0} \\ -1 \end{vmatrix}$$

or

$$-x - y - 2z = -1$$

$$-y + 2z = -1$$

$$\boxed{y = 2z + 1}$$

$$-x - 2z - \cancel{1} - 2z = -\cancel{1}$$

$$\boxed{x = -4z}$$

and the solutions are

$$\begin{cases} x = -4z \\ y = 2z + 1 \quad \forall z \in \mathbb{R} \\ z \end{cases}$$

2b. 10

$$\begin{array}{cccc} L & U & \doteq & A \end{array}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{vmatrix} \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{vmatrix}$$

dot row 1 of L with columns 1-3 of U to obtain:

$$\begin{vmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{vmatrix} \begin{vmatrix} 2 & -1 & -2 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{vmatrix}$$

dot row 2-3 of L with column 1 of U to obtain:

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & l_{32} & 1 \end{vmatrix} \begin{vmatrix} 2 & -1 & -2 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{vmatrix}$$

dot row 2 of L with columns 2-3 of U to obtain

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & l_{32} & 1 \end{vmatrix} \begin{vmatrix} 2 & -1 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & u_{33} \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{vmatrix}$$

dot row 3 of L with column 2 of U to obtain

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix} \begin{vmatrix} 2 & -1 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & u_{33} \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{vmatrix}$$

dot row 3 of L with column 3 of U to obtain

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix} \begin{vmatrix} 2 & -1 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ 4 & -2 & 8 \end{vmatrix}$$

$$\begin{array}{cccc} L & U & = & A \end{array}$$

$Ax = b \implies LUx = b \implies Lc = b$ where $c = Ux$

for $b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $c_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $x_1 = \begin{pmatrix} 9/4 \\ 5/6 \\ 4/3 \end{pmatrix}$

for $b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $c_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $x_2 = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/3 \end{pmatrix}$

for $b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $x_3 = \begin{pmatrix} 3/8 \\ 1/12 \\ 1/3 \end{pmatrix}$

let $X = \begin{vmatrix} \uparrow & \uparrow & \uparrow \\ X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{vmatrix} = \begin{vmatrix} 9/4 & 1/2 & 3/8 \\ 5/6 & 1/3 & 1/12 \\ 4/3 & 1/3 & 1/3 \end{vmatrix}$

$$AX = \begin{vmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{vmatrix} \begin{vmatrix} 9/4 & 1/2 & 3/8 \\ 5/6 & 1/3 & 1/12 \\ 4/3 & 1/3 & 1/3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I$$

since $AX = I$ $X = A^{-1} I = A^{-1}$

2c.

The characteristic polynomial of

$$A(t) = \begin{vmatrix} t & 1-t \\ 1 & 0 \end{vmatrix}$$

is $\det(A - \lambda I) = \det \begin{vmatrix} t-\lambda & 1-t \\ 1 & -\lambda \end{vmatrix} = 0$

$$-\lambda(t-\lambda) - (1-t) = 0$$

$$\lambda^2 - \lambda t + t - 1 = 0$$

$$\Delta = t^2 - 4(t-1) = t^2 - 4t + 4 = (t-2)^2$$

$$\lambda_{1,2} = \frac{t \mp (t-2)}{2} \quad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = t-1 \end{cases}$$

$\lambda_1 = 1$ does not depend on t

To find the eigen vector solve

$$\begin{pmatrix} t-1 & 1-t \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$x - y = 0 \qquad x = y$

which has solution

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for all t

~~3,1~~ 13

3 i. $E(X) = \sum k P_X(k) =$

$0(0,1) + 1(0,4) + 2(0,3) + 3(0,2) = 1,6$

ii. $Var(X) = E(X^2) - (E(X))^2 = E(X^2) - 1.6^2 =$

$E(X^2) = 0^2(0,1) + 1^2(0,4) + 2^2(0,3) + 3^2(0,2) =$
$= 3,4$

$Var(X) = 3.4 - 1.6^2 = 0,84$

iii. $E((X-2)^2) = (0-2)^2(0,1) + (1-2)^2(0,4) + (2-2)^2(0,3) +$
$(3-2)^2(0,2) = 1$

iv. $VAR(Y) = VAR((X-D)^2) = VAR(X^2 - 2XD + D^2) \underset{①}{=} VAR(X^2 - 2XD) =$
$\underset{②}{=} VAR(X^2) + VAR(-2DX) + 2\,COV(X^2, -2DX) = VAR(X^2) + 4D^2 VAR(X) +$
$+ (-4D)\,COV(X^2, X)$

① $VAR(X+C) = VAR(X)$, C – CONSTANT

② $VAR(X,Y) = VAR(X) + VAR(Y) + 2COV(X,Y)$

MINIMUM IS ACHIEVED WHEN

$\frac{\partial(VAR(Y))}{\partial D} = 0 \Rightarrow 8D\,VAR(X) = 4COV(X^2, X)$

$D = \frac{COV(X^2, X)}{2\cdot VAR(X)} = \frac{E(X^3) - E(X^2)E(X)}{2\cdot VAR(X)}$

$E(X^3) = 0^3 \cdot 0.1 + 1^3\, 0.4 + 2^3 \cdot 0.3 + 3^3\, 0.2 = 8.2$

$D = \frac{8.2 - 3.4 \cdot 1.6}{2 \cdot 0.84} = \frac{2.76}{1.68} \approx 1.64$

3(b)

(i) $V \sim \mathcal{U}(0,2) \Rightarrow V \leq 2 \Rightarrow \ln V \leq \ln 2 \Rightarrow \mathbb{P}(X \leq \ln 2) = 1$

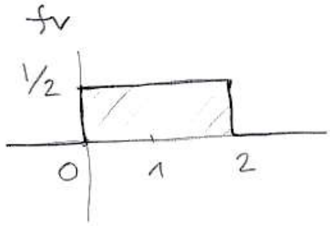
(ii) $F_X(x) = 1 \quad \forall x > \ln 2$

if $x \leq \ln 2$

$$F_X(x) = \mathbb{P}(\ln V \leq x) = \mathbb{P}(V < e^x) = F_V(e^x) =$$

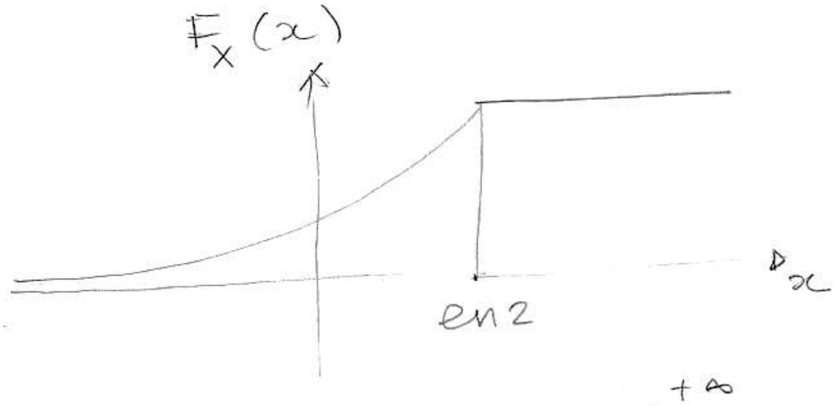
$$= \int_{-\infty}^{e^x} f_V(v)\, dV$$

$$f_V \begin{cases} 0 & V < 0 \\ 1/2 & 0 \leq V \leq 2 \end{cases}$$



$$= \int_0^{e^x} \frac{1}{2}\, dV = \frac{e^x}{2}$$

$$F_X(x) = \begin{cases} \dfrac{e^x}{2} & 0 < e^x \leq 2 \qquad x \leq \ln 2 \\ 1 & e^x > 2 \qquad x > \ln 2 \end{cases}$$



(iii) $E(X) = E(\ln V) = \int_{-\infty}^{+\infty} \ln V\, f_V\, dV =$

$$\frac{1}{2}\int_0^2 \ln V\, dV$$

do by parts

~~33~~ 15

$\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$uv - \int v \, du = x \ln x - \int \frac{x}{x} dx = x \ln x - x + C$

$\frac{1}{2} \int_0^2 \ln V \, dV = \frac{1}{2} \left[V \ln V - V \right]_0^2 =$

$V \ln V$ not defined in 0

However

$\lim_{V \to 0^+} V \ln V = \lim_{V \to 0^+} \frac{\ln V}{\frac{1}{V}}$

apply l'hopital

$= \lim_{V \to 0} \frac{\frac{1}{V}}{-\frac{1}{V^2}} = \lim_{V \to 0} -\frac{V^2}{V} = 0$

$\frac{1}{2} \left[V \ln V - V \right]_0^2 = \frac{1}{2} \left[2 \ln 2 - 2 \right] = \ln 2 - 1 \; (= -0{,}307)$

~~(iv)~~ $Var(X) = E(X^2) - (E(X))^2 =$

$= \int_{-\infty}^{+\infty} (\ln V)^2 f_V \, dV - (\ln 2 - 1)^2$

$= \frac{1}{2} \int_0^2 (\ln V)^2 dV - (\ln 2 - 1)^2$

again, do by parts

(16)

3.4

$\int (\ln x)^2 dx$

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x} dx \qquad v = x$

(*) This is the cumulative distribution of an exponential probability density function
$f(y,\lambda) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases}$

with rate parameter $\lambda = 1$

$\int u\,dv = uv - \int v\,du$

$\int (\ln x)^2 dx = x(\ln x)^2 - \int \cancel{x} \frac{2\ln x}{\cancel{x}} dx =$

$= x(\ln x)^2 - 2\int \ln x\, dx =$

$= x(\ln x)^2 - 2(x\ln x - x) + C$

$Var(X) = \frac{1}{2}\left[V(\ln V)^2 - 2V\ln V + 2V \right]_0^2 - (\ln 2 - 1)^2$

again $V(\ln V)^2$ not defined in $\emptyset$, but limit can be worked out as 0, by l'hopital.

$\frac{1}{2}\left[2(\cancel{\ln 2})^2 - \cancel{4\ln 2} + 4 - \cancel{2}(\cancel{\ln 2})^2 - 2 + \cancel{4\ln 2} \right] =$

$= \underline{1}$

(iv) $Y = \ln 2 - X \geq 0$ from (i) $P(X \leq \ln 2) = 1$

$y > 0$

$P(Y \leq y) = P(\ln 2 - \ln V \leq y) = P(-\ln \frac{V}{2} \leq y) =$

$= P(\ln \frac{V}{2} \geq -y) = P(\frac{V}{2} \geq e^{-y}) = 1 - P(\frac{V}{2} \leq e^{-y}) =$

$= 1 - P(V \leq 2e^{-y}) = 1 - \int_{-\infty}^{2e^{-y}} \frac{1}{2} dV = 1 - e^{-y}$

(*)