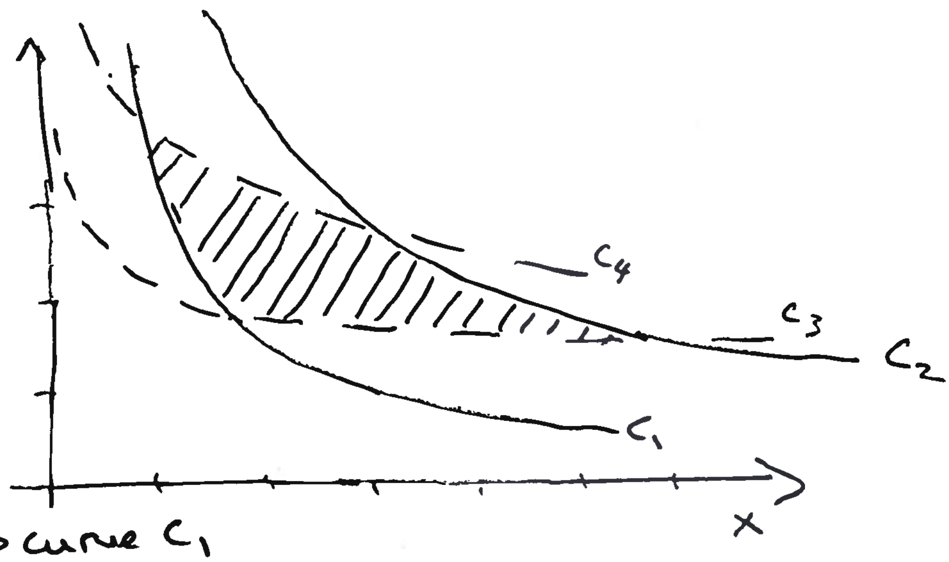
Q1 (a)



$xy = 1 \rightarrow$ curve C_1

$xy = 2 \rightarrow C_2$

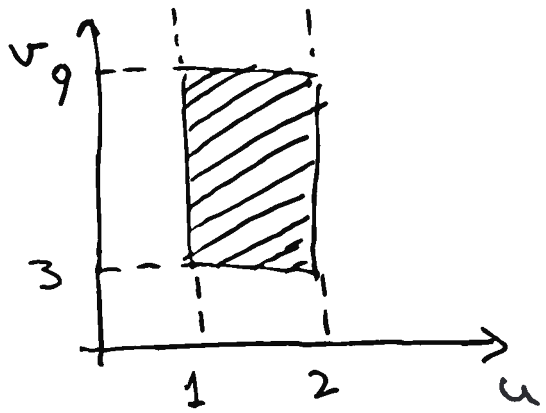
$xy^3 = 3 \rightarrow C_3$

$xy^3 = 9 \rightarrow C_4$

(b) Required area $A = \iint dx\,dy$ is difficult to integrate directly, as limits are quite complicated. We can use a transformation:

$u = xy \quad , \quad v = xy^3$

$\Rightarrow$ Area becomes $\displaystyle\int_3^9 \int_1^2 \frac{1}{J}\, du\, dv$



J: Jacobian

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} = 2xy^2 = 2v$$

Therefore required area is

$$\int_3^9 \int_1^2 \frac{1}{2v}\,du\,dv = [u]_1^2 \cdot \int_3^9 \frac{1}{2v}\,dv$$

$$= (2-1)\cdot\frac{1}{2}\cdot \ln\frac{9}{3} = \underline{\underline{\frac{1}{2}\ln 3}}\,.$$

(c) $\oint \underline{F}\cdot d\underline{l} = \iint_R \nabla\times\underline{F}\; dA$ by Stokes Theorem.

$$\underline{F} = \frac{2}{5}(xy^5)^{1/2}\underline{i} + (xy)^{3/2}\underline{j}$$

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2}{5}(xy^5)^{1/2} & (xy)^{3/2} & 0 \end{vmatrix} = -\frac{\partial}{\partial y}\left(\frac{2}{5}x^{1/2}y^{5/2}\right) + \frac{\partial}{\partial x}\left(x^{3/2}y^{3/2}\right)$$

$$= -x^{1/2}y^{3/2} + \frac{3}{2}x^{1/2}y^{3/2} = \frac{1}{2}(xy^3)^{1/2}$$

Integration is best done using the transformation of part (b)

$$\therefore \text{RHS} = \int_3^9\int_1^2 \left(\frac{1}{2}\right)\frac{v^{1/2}}{2v}\,du\,dv$$

$\Rightarrow$ Required integral is

$$-\frac{1}{4}\int_3^9 v^{-1/2}\,dv \int_1^2 du = \left(-\frac{1}{4}\right)\cdot\left(2\left[v^{1/2}\right]_3^9\right)\cdot 1$$

$$= \cdot\frac{1}{2}\left(3-\sqrt{3}\right)$$

NOTES: Part (b) could also be done by direct integration, taking care of the various limits & curves. But in comparison to the transformation, it is quite laborious & prone to error.

Q2(a) The divergence theorem is written as

$$\int_{\text{volume } V} \nabla \cdot \underline{F} \, dV = \oint_{\substack{\text{surface } S \\ \text{enclosing } V}} \underline{F} \cdot \underline{n} \, dA$$

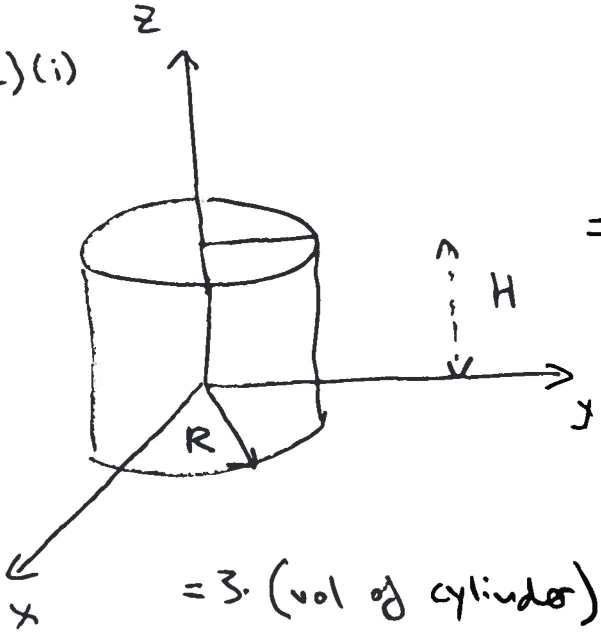
It is valid for any volume V enclosed by a closed surface S, and for any vector field $\underline{F}$, provided it is differentiable.

(b) The group $\phi \nabla \psi$ can also be thought of as a vector field. Hence if $\underline{F} = \phi \nabla \psi$ and using the divergence theorem, we get

$$\int \nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla \cdot \nabla \psi$$
$$= \nabla \phi \nabla \psi + \phi \nabla^2 \psi$$

$\therefore$ Green's identity is shown.

(c)(i)



$$\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$\Rightarrow \nabla \cdot \underline{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$
$$= 3$$

Hence $\oint \underline{F} \cdot \underline{n} \, dA = \int_V 3 \, dV$

$= 3 \cdot (\text{vol of cylinder}) = \underline{\underline{3\pi R^2 H}}$.

(c)(ii) We evaluate $\oint \underline{F} \cdot \underline{n} \, dA$ directly.

For the top surface $(z = H)$, $\underline{n}$ being pointing outwards z normal to the surface implies

$\underline{n} = \underline{k} \Rightarrow \underline{F} \cdot \underline{n} = z$ at that location, i.e. H.

Hence $$\int_{\substack{\text{top circular}\\ \text{face}}} \underline{F} \cdot \underline{n} \, dA = H \int dA = H \pi R^2$$

For the lower circular face at $z = 0$, $\underline{n} = -\underline{k}$

$\underline{F} \cdot \underline{n} = -z = 0$. Hence, no contribution.

On the curved surface, the vector normal to the surface is the unit vector in the direction

$x\underline{i} + y\underline{j}$, i.e. $\dfrac{x\underline{i} + y\underline{j}}{\sqrt{x^2+y^2}} = \dfrac{x\underline{i} + y\underline{j}}{R}$

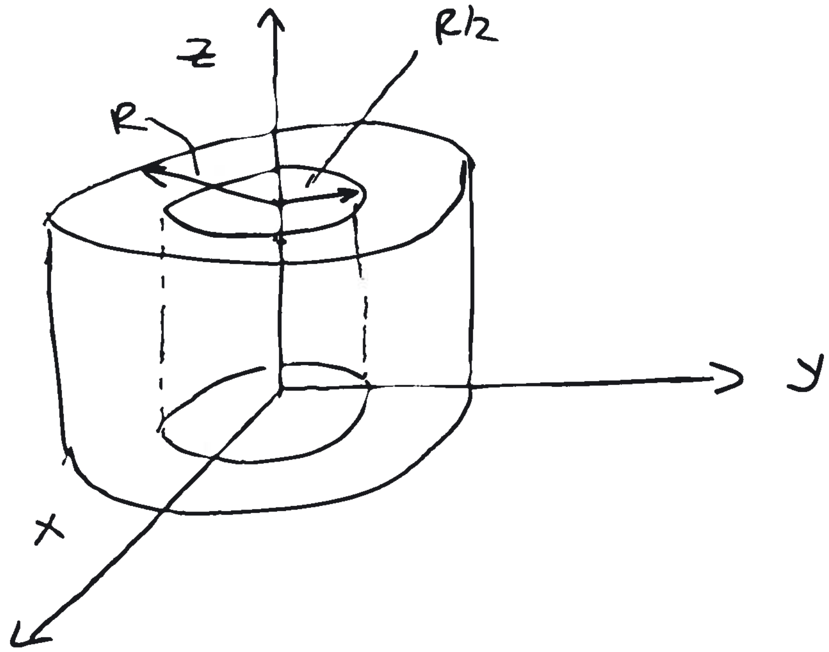
$\underline{F} \cdot \underline{n}$ now becomes $\dfrac{x^2 + y^2}{R} = \dfrac{R^2}{R} = R$

Hence $$\int_{\substack{\text{curved}\\ \text{surface}}} \underline{F} \cdot \underline{n} \, dA = R \int dA = R \cdot 2\pi R H$$
$$= 2\pi R^2 H$$

Therefore total flux of $\underline{F}$ out of the cylinder is $\pi R^2 H + 2\pi R^2 H = 3\pi R^2 H$, as before.

Divergence theorem is demonstrated.

(c)(iii)

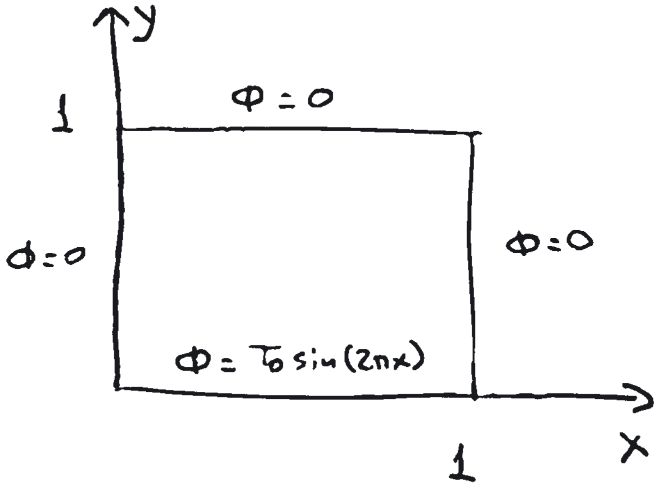


From the result of Part (b), the flux of $\underline{F}$ leaving a cylinder of height H & radius R is $3\pi R^2 H$. So, for the smaller cylinder, this is $\frac{3\pi}{4} R^2 H$. The contribution to this from the top circular face is $\pi H \left(\frac{R}{2}\right)^2 = \frac{1}{4}\pi R^2 H$.
From the curved surface, it is $2\pi H \left(\frac{R}{2}\right)^2 = \frac{2}{4}\pi R^2 H$
So, considering the new (tyre-looking) shape, the total flux of $\underline{F}$ outwards from the shape is

$$\underbrace{(3\pi R^2 H)}_{\text{from Part (c)(ii) if it were solid with radius } R} - \underbrace{\left(\tfrac{1}{4}\pi R^2 H\right)}_{\text{missing top surface, no flux contribution}} - \underbrace{\left(\tfrac{2}{4}\pi R^2 H\right)}_{\text{from the curved surface at } r = R/2 \text{ (-ve because it comes in)}} = \frac{9}{4}\pi R^2 H.$$

NOTE: It can also be done using divergence theorem on remaining volume.

Q3 (a)



If $\Phi = T - T_0$, then the equation for $f(x,y) = 0$ becomes again $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$

Separation of variables means we seek a solution

$$\Phi(x,y) = X(x)\, Y(y) \Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} = c$$

where c is a constant.

Let us consider the b.c.'s first.

At $x = 1$, $\Phi = 0 \Rightarrow \Phi(1,y) = X(1) \cdot Y(y) = 0$

To avoid the trivial solution, it must be that $X(1) = 0$.

At $x = 0$, $\Phi = 0$. Similarly, $X(0) = 0$.

At $y = 1$, $\Phi = 0$. Similarly, $Y(1) = 0$.

At $y = 0$, $\Phi(x,0) = T_0 \sin(2\pi x)$

$\Rightarrow X(x) = T_0 \sin(2\pi x)$ & $Y(0) = 1$.

Writing the separation constant as $-k^2$ $(k>0)$ we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \quad \& \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = k^2$$

We suspect a sin dependence of X due to the boundary condition before, so it helps to keep the $-ve$ in that part. Therefore, if

$$X = T_0 \sin(2\pi x), \quad k = 2\pi$$

Hence, $Y = Ae^{2\pi y} + Be^{-2\pi y}$

At $y=0$, $Y=1$ $\Rightarrow$ $A+B=1$

At $y=1$ $Y=0$ $\Rightarrow$ $Ae^{2\pi} + Be^{-2\pi} = 0$

Manipulating these gives

$$A = -\frac{e^{-2\pi}}{e^{2\pi}-e^{-2\pi}} \quad \& \quad B = \frac{e^{2\pi}}{e^{2\pi}-e^{-2\pi}}$$

Final result: $\Phi = T_0 \sin(2\pi x)\left(Ae^{2\pi y} + Be^{-2\pi y}\right)$

$$\Rightarrow T = T_0\left[1 + \sin(2\pi x)\left(Ae^{2\pi y} + Be^{-2\pi y}\right)\right]$$

(b) Since $f \neq 0$, our previous solution does not hold any more. But we can progress using Gauss theorem.

$$\iint \nabla \cdot \underline{q} \, dA = \oint \underline{q} \cdot d\underline{n}$$

over all 2D domain

over bounding curve

dA: area element

$\underline{n}$: vector normal to bounding curve

$$\underline{q} = -\nabla T \Rightarrow \nabla \cdot \underline{q} = -\nabla^2 T$$

$\Rightarrow$ Required total flux $= \int_0^1 \int_0^1 \underbrace{-\nabla^2 T}_{= f(x,y) \text{ from the pde}} \, dx \, dy$

$$= -\int_0^1 \sin(4\pi x)\, dx \int_0^1 \sin(4\pi y)\, dy$$

$$= \frac{1}{4\pi} \left[\cos(4\pi x) \right]_0^1 \left(-\frac{1}{4\pi} \right) \left[\cos(4\pi y) \right]_0^1$$

$$= 0$$

So the net flux of heat in and out is zero. From an energy conservation perspective, this is something we could have guessed given the symmetry about $x = 0.5$ for the b.c. at $y = 0$ but also the source term.

Q4. 1/6

(a)

$$\begin{bmatrix} 3 & 4 & 0 & 1 \\ 6 & 4 & 2 & t \\ 9 & 0 & 6 & 1 \end{bmatrix} - \begin{array}{c|cccc} & 3 & 4 & 0 & 1 \\ \hline 1 & 3 & 4 & 0 & 1 \\ 2 & 6 & 8 & 0 & 2 \\ 3 & 9 & 12 & 0 & 3 \end{array} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & t-2 \\ 0 & -12 & 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & t-2 \\ 0 & -12 & 6 & -2 \end{bmatrix} - \begin{array}{c|cccc} & 0 & -4 & 2 & t-2 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 2 & t-2 \\ 3 & 0 & -12 & 6 & 3t-6 \end{array} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4-3t \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 4 & 0 & 1 \\ 0 & -4 & 2 & t-2 \\ 0 & 0 & 0 & 4-3t \end{bmatrix}$$

IF $t = \frac{4}{3}$ THEN A IS OF RANK 2.

(G)

$$A = \begin{vmatrix} 1 & k+1 & 1 \\ k-3 & 1 & 1 \\ 1 & 1 & 1-k \end{vmatrix} \quad b = \begin{vmatrix} k+5 \\ 1 \\ 5 \end{vmatrix}$$

$$\det|A| = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1-k \end{vmatrix} - (k+1) \begin{vmatrix} k-3 & 1 \\ 1 & 1-k \end{vmatrix} + 1 \begin{vmatrix} k-3 & 1 \\ 1 & 1 \end{vmatrix} =$$

$$= 1-k-1 - (k+1)\big((k-3)(1-k) - 1\big) + k-3-1 =$$

$$= -4 - \big((k-3)(1-k^2) - 1 - k\big) = -4 + k + 1 - k(1-k^2) +$$
$$+ 3(1-k^2) = k - k(1-k^2) - 3k^2 = k^3 - 3k^2 = k^2(k-3)$$

IF $k \neq 0$ AND $k \neq 3$ THEN SOLUTION IS UNIQUE. FIND BY GAUSSIAN ELIMINATION.

$$\begin{vmatrix} 1 & k+1 & 1 \\ k-3 & 1 & 1 \\ 1 & 1 & 1-k \end{vmatrix} \begin{vmatrix} k+5 \\ 1 \\ 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & k & k \\ k-3 & 1 & 1 \\ 1 & 1 & 1-k \end{vmatrix} \begin{vmatrix} k \\ 1 \\ 5 \end{vmatrix} \Rightarrow$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ k-3 & 1 & 1 \\ 1 & 1 & 1-k \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ k-3 & 0 & 0 \\ 1 & 1 & 1-k \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ k-3 & 0 & 0 \\ 1 & 0 & -k \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 4 \end{vmatrix}$$

HENCE $x = 0$, $z = -\frac{4}{k}$, $y = 1 + \frac{4}{k}$

(6 - CONTINUED)

IF $k=0$ THEN

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 5 \\ 1 \\ 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 5 \\ -4 \\ 0 \end{vmatrix}$$

HENCE THERE IS A FAMILY OF SOLUTIONS:

$x = 1,\ y = 4 - z$

IF $k=3$ THEN

$$\begin{vmatrix} 1 & 4 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix} \begin{vmatrix} 8 \\ 1 \\ 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 3 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix} \begin{vmatrix} 3 \\ 1 \\ 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 4 \end{vmatrix}$$

HENCE THERE IS A FAMILY OF SOLUTIONS:

$x = 4 + 3z,\ y = 1 - z$

(c) $C - 2I = -C^2$

LET λ, x BE ANY EIGEN VALUE AND EIGEN VECTOR.

WE HAVE $Cx = \lambda x$. HENCE

$$(C-2I)x = -C^2 x$$

$$\Downarrow$$

$$Cx - 2Ix = -C^2 x$$

$$\Downarrow$$

$$\lambda x - 2x = -C(\lambda x)$$

$$\Downarrow$$

$$(\lambda - 2)x = -\lambda^2 x$$

$$(\lambda^2 + \lambda - 2)x = 0$$

HENCE $\lambda = \dfrac{-1 \pm \sqrt{1+8}}{2} = \underline{1} \text{ OR } -2$

(a)

$$A = \begin{bmatrix} a & b & c \\ c & a & 0 \\ c & 0 & a \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b & b \\ c & a-\lambda & 0 \\ c & 0 & a-\lambda \end{bmatrix}$$

$$\det(A - I\lambda) = (a-\lambda)\begin{vmatrix} a-\lambda & 0 \\ 0 & a-\lambda \end{vmatrix} - b\begin{vmatrix} c & 0 \\ c & a-\lambda \end{vmatrix}$$

$$+ b\begin{vmatrix} c & a-\lambda \\ c & 0 \end{vmatrix} = (a-\lambda)^3 - 2bc(a-\lambda)$$

$$- bc(a-\lambda) = (a-\lambda)((a-\lambda)^2 - 2bc)$$

$$\begin{bmatrix} a & b & b \\ c & a & 0 \\ c & 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

EIGEN VECTOR FOR EIGENVALUE a

$$\therefore ax + by + bz = ax \quad \Rightarrow$$

$$cx + ay + 0 = ay \quad \Rightarrow \quad x = 0 \text{ AND } y = -z$$

$$cx + 0y + az = az \quad \Rightarrow$$

(a - CONTINUED)

OTHER EIGEN VALUES ARE:

$$\lambda = a \pm \sqrt{2bc}$$

$$\begin{bmatrix} a & b & b \\ c & a & 0 \\ c & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (a \pm \sqrt{2bc}) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$ax + by + bz = ax \pm \sqrt{2bc}\, x \Rightarrow by + bz = \pm\sqrt{2bc}\, x$$

$$cx + ay = (a \pm \sqrt{2bc})y \Rightarrow y = \pm \frac{cx}{\sqrt{2bc}}$$

$$cx + az = (a \pm \sqrt{2bc})z \Rightarrow z = \pm \frac{cx}{\sqrt{2bc}}$$

$$by + bz = \pm \frac{bcx}{\sqrt{2bc}} + \pm \frac{bcx}{\sqrt{2bc}} = \pm\sqrt{2bc}\, x$$

$$\Downarrow$$

$$\pm \frac{2\sqrt{bc}}{\sqrt{2}} = \pm\sqrt{2bc}\, x$$

CORRECT FOR ALL x.

SO EIGEN VECTORS ARE $\begin{bmatrix} \pm\sqrt{2b/c} \\ 1 \\ 1 \end{bmatrix}$.

FOR ANY n: $\quad \mathrm{DET}(A) = (a-\lambda)^{n-2}\left((a-\lambda)^2 - (n-2)bc\right)$

EIGENVECTORS: $\begin{bmatrix} \pm\sqrt{(n-2)b/c} \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $\underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ -1 \end{bmatrix}}_{n-2}$

Q5 (a)(i)

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$q_2 = \frac{\sqrt{2}}{\sqrt{5}} \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \quad \tilde{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5} \left(\begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ -\frac{1}{5} \\ \frac{3}{5} \end{bmatrix}$$

$$q_3 = \frac{1}{\sqrt{15}} \begin{bmatrix} -1 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{15}} \\ 0 & \frac{\sqrt{2}}{\sqrt{5}} & -\frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{15}} \\ 0 & \frac{\sqrt{2}}{\sqrt{5}} & \frac{3}{\sqrt{15}} \end{bmatrix} \quad R = \overset{Q^T}{\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{\sqrt{2}}{\sqrt{5}} & \frac{1}{\sqrt{10}} & \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{bmatrix}} \overset{A}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} =$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{5}} \\ 0 & 0 & \frac{3}{\sqrt{15}} \end{bmatrix}$$

(a-ii)

A - RANK 3

$$A^T A x = A^T b$$

$$(QR)^T QRx = (QR)^T b$$

HENCE $\quad R^T Q^T Q R x = R^T Q^T b$;

$$R^T R x = R^T Q^T b \qquad (Q - \text{ORTHOGONAL})$$

$$R x = Q^T b \qquad (R - \text{NONSINGULAR})$$

$$\begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{5}} \\ 0 & 0 & \frac{3}{\sqrt{15}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{\sqrt{2}}{\sqrt{5}} & \frac{1}{\sqrt{10}} & \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{1}{\sqrt{15}} & -\frac{2}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \frac{3}{\sqrt{15}} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$z = \frac{\sqrt{15}}{3} \left(\frac{1 - 2 \cdot 0 - 1 + 6}{\sqrt{15}} \right) = 2$$

$$\frac{\sqrt{5}}{\sqrt{2}} y + \frac{2\sqrt{2}}{\sqrt{5}} = -\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{10}} + \frac{2\sqrt{2}}{\sqrt{5}} \Rightarrow y = 0$$

$x = 1$

SO $\quad x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

(6) $$A = \begin{pmatrix} a & 0 \\ 1 & -a \end{pmatrix}$$

WHEN $a \neq 0$ EIGEN VALUES ARE DISTINCT: a AND $-a$ (DIAGONAL) SINCE A IS LOWER TRIANGULAR

EIGEN VECTORS ARE: FOR $\lambda = a$

$$Ax = ax \Rightarrow \begin{bmatrix} a x_1 \\ x_1 - a x_2 \end{bmatrix} = \begin{bmatrix} a x_1 \\ a x_2 \end{bmatrix} \Rightarrow x_2 = \frac{1}{2a} x_1 \Rightarrow \begin{bmatrix} 2a \\ 1 \end{bmatrix} \frac{1}{\sqrt{4a^2+1}}$$

FOR $\lambda = -a$

$$\begin{bmatrix} a & 0 \\ 1 & -a \end{bmatrix} x = -ax \Rightarrow \begin{bmatrix} a x_1 \\ x_1 - a x_2 \end{bmatrix} = \begin{bmatrix} -a x_1 \\ -a x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A CAN BE DIAGONALISED AS

$$D = P^{-1} A P, \text{ WHERE } D = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \; P = \begin{bmatrix} \frac{2a}{\sqrt{4a^2+1}} & 0 \\ \frac{1}{\sqrt{4a^2+1}} & 1 \end{bmatrix}$$

HENCE $A = PDP^{-1}$. SO $A^{20} = PD^{20}P^{-1} =$

$$= P \begin{bmatrix} a & \\ & -a \end{bmatrix}^{20} P^{-1} = a^{20} P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = a^{20} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

IF $a = 0$ THEN

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ HENCE } A^{20} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

① SINCE X_1 AND X_2 ARE ON A PLANE.

(c) NORMAL OF A PLANE IS $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

SINCE FOR ANY TWO POINTS $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

WE HAVE A VECTOR $\vec{v} = X_1 - X_2$.

$$\vec{n} \cdot \vec{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix} = 2x_1 - 3y_1 + z_1 - (2x_2 - 3y_2 + z_2) \underset{①}{=} 0 - 0 = 0$$

(C - CONTINUED)

LET $u = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ AND $v = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ TWO LINEARLY INDEPENDENT POINTS ON THE PLANE.

USING GRAHAM-SCHMIDT:

$$\vec{q}_1 = \frac{u}{\|u\|} = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\tilde{v} = v - (q_1 v) q_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{\sqrt{13}} \left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right) \cdot \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - \frac{2}{13} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -6 \\ 9 \\ 39 \end{bmatrix} = \frac{3}{13} \begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix}$$

$$\vec{q}_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix}$$

SO THE BASIS IS $\left\{ \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{182}} \begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix} \right\}$

$$q_1 \cdot q_2 \propto \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix} = -6 + 6 = 0$$

$$\vec{q}_1 \cdot \vec{n} \propto \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$\vec{q}_2 \, \vec{n} \propto \begin{bmatrix} -2 \\ 3 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = -4 - 9 + 13 = 0$$

(d) (i)

$$A^TA = \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EIGEN VALUES ARE $1, a^2$ AND 0 (DIAGONAL).

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SINGULAR VALUES: $1, |a|, 0$

$$\hat{q}_1 = \frac{A q_1}{1} = \frac{\begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{q}_2 = \frac{A q_2}{|a|} = \frac{\begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{|a|} = \frac{1}{|a|} \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}$$

EIGENVECTOR FOR EIGENVALUE 0 IS $\begin{bmatrix} 1 \\ 0 \\ -a \end{bmatrix} \cdot \frac{1}{\sqrt{1+a^2}}$

(d-i - CONTINUED)

IF $0<|a|<1$ THEN

$$A=\hat{Q}\Sigma Q^T=\begin{bmatrix} 0 & \frac{a}{|a|} & 1\cdot\frac{1}{\sqrt{1+a^2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{|a|} & -a\cdot\frac{1}{\sqrt{1+a^2}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & |a| & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=$$

$$=\begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

IF $|a|>1$

$$A=\hat{Q}\Sigma Q^T \begin{bmatrix} \frac{a}{|a|} & 0 & 1\cdot\frac{1}{\sqrt{1+a^2}} \\ 0 & 1 & 0 \\ \frac{1}{|a|} & 0 & -a\frac{1}{\sqrt{1+a^2}} \end{bmatrix}\begin{bmatrix} |a| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}=$$

$$=\begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

IF $a=0$ THEN $\hat{q}_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\hat{q}_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ AND EIGENVECTOR FOR $\lambda=0$ OF AA^T IS $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

HENCE $A=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(ii) COLUMN SPACE $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix}\right\}$ ROW SPACE $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

LEFT NULL SPACE $\left\{\begin{bmatrix} 1 \\ 0 \\ -a \end{bmatrix}\right\}$ NULL SPACE $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

Q6.

(a)(i)

FOR BOTH TEAMS TO PLAY ALL 7 ROUNDS, BOTH OF THEM NEED TO WIN 3 TIMES.

$$P(\text{A WINS 3 OF 6}) = \binom{6}{3} p^3(1-p)^3 = \frac{6!}{3!\,3!} p^3(1-p)^3 = 20\, p^3(1-p)^3$$

NOTE, WHEN A WINS 3 THEN B WINS 3 AS WELL.

(a)(ii)

LET k BE NUMBER OF HOME WINS FOR A.

THEN

$$P(\text{A WINS 3 OF 6}) = \sum_{k=0}^{3} \binom{3}{k} p^k (1-p)^{3-k} \cdot \binom{3}{k} q^{3-k} (1-q)^k$$

(b)

LET X^0 AND X^1 BE RANDOM VARIABLES COUNTING THE NUMBER OF TIMES 0s AND 1s TRANSMITTED RESPECTIVELY.

$Y = X^0 + X^1$ – TOTAL NUMBER OF SYMBOLS TRANSMITTED.

WE HAVE:

$$P(X^0=n, X^1=m) = P(X^0=n, X^1=m \mid Y=n+m)\, P(Y=n+m) =$$

$$= \binom{n+m}{n} p^n (1-p)^m \frac{\lambda^{n+m} e^{-\lambda}}{(n+m)!} = \frac{(p\lambda)^n e^{-\lambda p}}{n!} \cdot \frac{e^{-\lambda+\lambda p} (\lambda(1-p))^m}{m!}$$

HENCE $P(X^0=n) = \sum_{m=0}^{\infty} P(X^0=n, X^1=m) =$

(e - CONTINUED)

$$= \frac{e^{-\lambda p}(\lambda p)^n}{n!} e^{-\lambda(1-p)} \sum_{m=0}^{\infty} \frac{\lambda(1-p)^m}{m!} \overset{(1)}{=}$$

$$= \frac{e^{-\lambda p}(\lambda p)^n}{n!} e^{-\lambda(1-p)} e^{\lambda(1-p)} = \frac{e^{-\lambda p}(\lambda p)^n}{n!}$$

(1) - POISSON PMF WITH PARAMETER $\lambda(1-p)$ SUMS TO 1.

HENCE X^0 IS POISSON WITH PARAMETER λp.

(c) (i)

$$F_Z(z) = P(Z \le z) = p P(X \le z) + (1-p) P(Y \le z) = p F_X(z) +$$
$$+ (1-p) F_Y(z)$$

DIFFERENTIATING WITH RESPECT TO z, WE GET:

$$f_Z(z) = p f_X(z) + (1-p) f_Y(z)$$

(ii) LET X BE RV WITH PDF:

$$f_X(x) = \begin{cases} \lambda e^{\lambda x}, & \text{IF } x < 0, \\ 0, & \text{OTHERWISE} \end{cases}$$

AND Y s.t.

$$f_Y(y) = \begin{cases} 0 & \text{IF } y < 0 \\ \lambda e^{-\lambda y} & \text{OTHERWISE} \end{cases}$$

CDF'S OF X AND Y ARE

$$F_X(x) = \begin{cases} e^{\lambda x}, & \text{IF } x<0 \\ 1 & \text{OTHERWISE} \end{cases}$$

AND

$$F_Y(y) = \begin{cases} 0, & \text{IF } y<0 \\ 1-e^{-\lambda y}, & \text{OTHERWISE} \end{cases}$$

FROM PART (C-i) WE HAVE

$$F_Z(z) = p F_X(z) + (1-p) F_Y(z) =$$

$$= \begin{cases} pe^{\lambda z}, & z<0 \\ p+(1-p)(1-e^{-\lambda z}) = 1-(1-p)e^{-\lambda z}, & \text{OTHERWISE} \end{cases}$$

(C-iii)

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\, dz = \int_{-\infty}^{0} z p \lambda e^{\lambda z}\, dz + \int_{0}^{\infty} z(1-p)\lambda e^{-\lambda z}\, dz$$

$$= -p\int_{0}^{\infty} z\lambda e^{-\lambda z}\, dz + (1-p)\int_{0}^{\infty} z\lambda e^{-\lambda z}\, dz = -\frac{p}{\lambda} + \frac{(1-p)}{\lambda} = \frac{1-2p}{\lambda}$$

$$E(Z^2) = \int_{-\infty}^{\infty} z^2 f_Z(z)\, dz = \int_{-\infty}^{0} z^2 p\lambda e^{\lambda z}\, dz + \int_{0}^{\infty} z^2(1-p)\lambda e^{-\lambda z}\, dz =$$

$$= \frac{2p}{\lambda^2} + \frac{2(1-p)}{\lambda^2} = \frac{2}{\lambda^2} \quad \text{HENCE } \operatorname{VAR}(Z) = \frac{2}{\lambda^2} - \frac{(1-2p)^2}{\lambda^2}$$

(d) 41%

$$P(S<s) = P(\max(X_1, X_2, \dots X_n)) =$$

$$= P(X_1<s \cap X_2<s \dots \cap X_n<s) \underset{①}{=} P(X<s)^n \underset{②}{=} (1-e^{-\lambda s})^n$$

① INDEPENDENCE OF X,Y,Z

② CDF OF EXP(λ)

③ DERIVATIVE OF CDF W.R.T. S.

HENCE PDF $f_S(s) = n \lambda e^{-\lambda s} (1-e^{-\lambda s})^{n-1}$ ③