3A1

2017

MASTER CRIB

WMD
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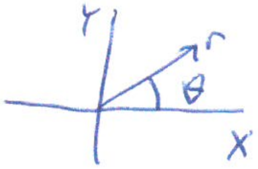
(a) $\phi = A r^{\alpha} \cos\alpha\theta$: $\dfrac{\partial\phi}{\partial r} = A\alpha r^{\alpha-1}\cos\alpha\theta$; $\dfrac{\partial^2\phi}{\partial r^2} = A\alpha(\alpha-1) r^{\alpha-2}\cos\alpha\theta$



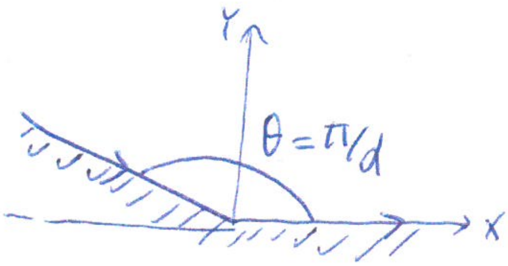
$$\frac{\partial^2\phi}{\partial\theta^2} = -A\alpha^2 r^{\alpha}\cos\alpha\theta$$

$$\therefore \quad \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{A}{r^2}\Big[\underbrace{\alpha(\alpha-1) + \alpha - \alpha^2}_{\cancel{\alpha^2} - \cancel{\alpha} + \cancel{\alpha} - \cancel{\alpha^2}}\Big] r^{\alpha}\cos\alpha\theta$$

$= 0$ $\therefore$ ϕ is valid potential

(b) Still in polar coordinates: $u_r = \dfrac{\partial\phi}{\partial r} = A\alpha r^{\alpha-1}\cos\alpha\theta$

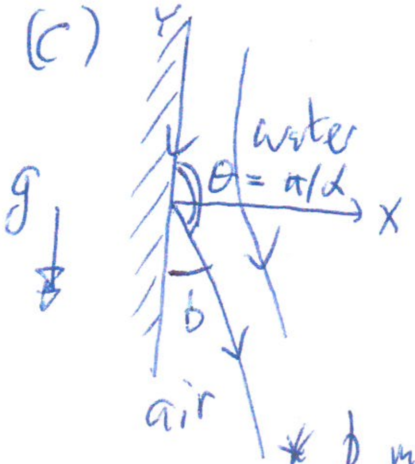
$u_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -A\alpha r^{\alpha-1}\sin\alpha\theta$



$u_\theta = 0$, streamline, if $\alpha\theta = n\pi$

eg. corner flow, $\theta = \pi/\alpha$

(c)



Assume the potential flow above and approximate the bubble as the corner flow.

Bernoulli: $p_0 = p + \rho g \underbrace{r\cos\theta}_{\text{vertical height}} + \frac{1}{2}\rho(\underbrace{A\alpha r^{\alpha-1}\cos\alpha\theta}_{u_r})^2$

$u_\theta = 0$

* p must be uniform as the bubble is constant pressure $\Rightarrow$ $\rho g r\cos\theta + \frac{1}{2}\rho A^2\alpha^2 r^{2(\alpha-1)}\cos^2\alpha\theta$ = constant

This can only be independent of r if the "r" terms cancel, i.e. $r = r^{2(\alpha-1)}$

$$\therefore 2(\alpha-1) = 1 \Rightarrow \alpha = 3/2$$

$$\therefore \theta = 2\pi/3 \ \& \ \underline{b = \pi/3}$$

(d) Hence:

$$r[\cancel{\rho} g\cos\theta + \tfrac{1}{2}\cancel{\rho} A^2\alpha^2\cos^2\alpha\theta] = \text{constant}$$

[ ] = 0 for uniform flow w.r.t. r

$$\therefore \underbrace{g}_{9.81}\underbrace{\cos\tfrac{2\pi}{3}}_{\sqrt{3}/2} + \tfrac{1}{2}A^2(\cancel{3}/2)^2 \underbrace{\cos^2\tfrac{\cancel{3}}{2}\cdot\tfrac{2\pi}{\cancel{3}}}_{(-1)^2} = 0$$

$\cos 0 = 1$

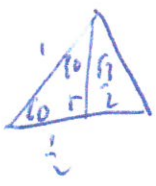
$\cos\frac{\pi}{2} = 0$ ~~cos 2π/3~~

$\cos\pi = -1$

$\cos 120$

$= -\cos 30$

$= \sqrt{3}/2$

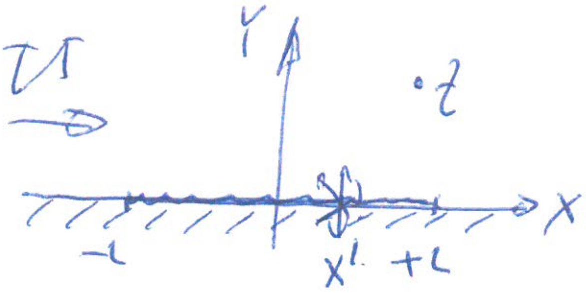


$$8.495 + 1.125 A^2 = 0$$

$$\therefore \underline{A = 2.75?}$$

For upstream & downstream the flow will be uniform.

(2)



(a) Porous section modelled via a series of sources with local strength $-2V\Delta x$

extracts from one side only

vol. flow per unit length

Associated complex potential $\Rightarrow$ $\dfrac{-2V\Delta x}{2\pi}\log(z-x')$

$\therefore$ overall flow, $F(z) = Uz - \displaystyle\int_{-L}^{+L}\frac{V}{\pi}\log(z-x')\,dx'$

uniform flow.

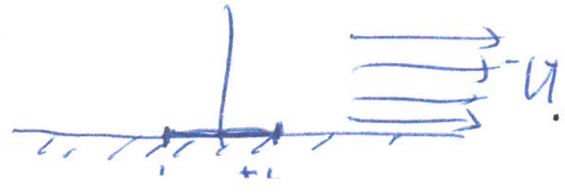
(b) Hence, velocity field $\dfrac{dF}{dz} = U - \displaystyle\int_{-L}^{+L}\frac{V}{\pi}\frac{1}{z-x'}\,dx'$

$$= U + \frac{V}{\pi}\Big[\log(z-x')\Big]_{-L}^{+L}$$

$$= U + \frac{V}{\pi}\log\left(\frac{z-L}{z+L}\right)$$

(i) $z \gg L$: $\log\left(\frac{z-L}{z+L}\right) \to \log 1 = 0$ $\therefore$ $\dfrac{dF}{dz} = u - iv \to U$

uniform flow

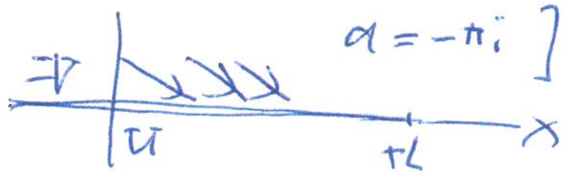


(ii) $z << L$ but $z > 0$: $\log\left(\frac{z-L}{z+L}\right) \to \log -1 = i\pi$

$$\therefore \frac{dF}{dz} = u - iv \to U - iV$$

[$a = \log -1$, $e^{a} = -1$
$\cos a + i \sin a = -1$
$a = -\pi i$]

i.e. uniform flow at angle $\tan^{-1}(V/U)$



(c) Stagnation points, $\frac{dF}{dz} = 0$ $\therefore$ $U = -\frac{V}{\pi}\log\left(\frac{z-L}{z+L}\right)$

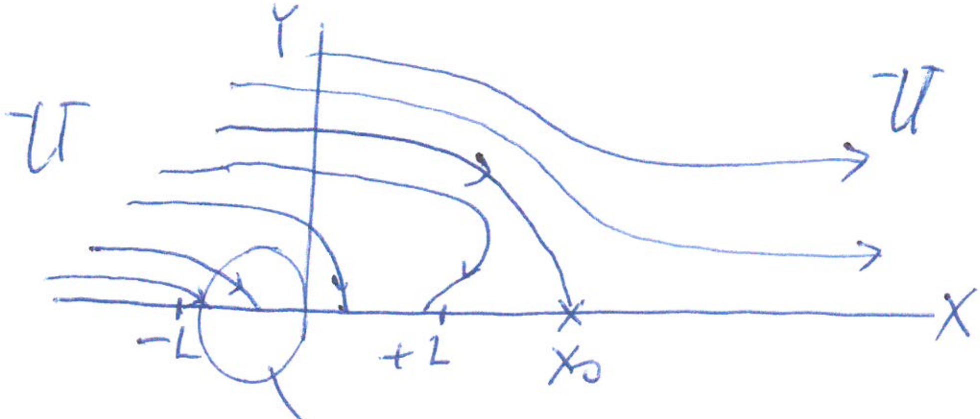
by symmetry they will lie on the ~~X~~-axis so $y=0$

Set $z = x_0$: $\frac{x_0 - L}{x_0 + L} = e^{-\pi U/V}$

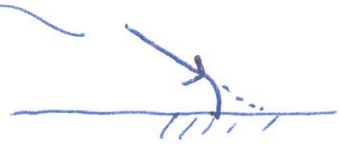
$$x_0(1 - e^{-\pi U/V}) = L(1 + e^{\cdots})$$

$$\therefore \underline{\frac{x_0}{L} = \frac{1 + e^{-\pi U/V}}{1 - e^{-\pi U/V}}} > 1$$

(d) Hence:



(e) with viscosity, the stream-lines exit at 90°.



③ (a) A flow has potential $F(z) = U(z - ae^{ikz})$

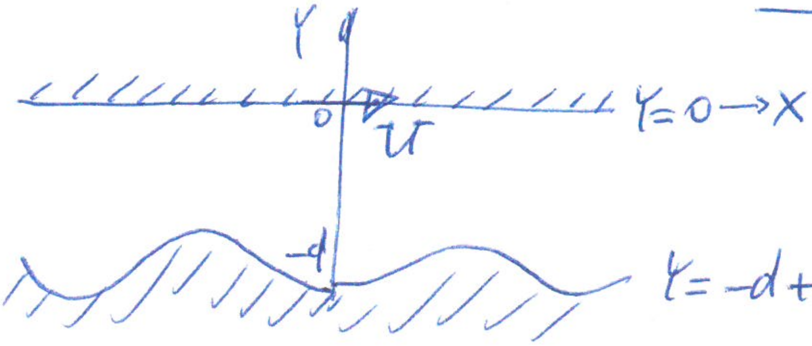
$$\therefore F(z) = U(x+iy) - Uae^{ik(x+iy)}$$
$$= U(x+iy) - Uae^{ikx}e^{-ky} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$= U\{(x - ae^{-ky}\cos kx) + i(y - ae^{-ky}\sin kx)\}$$
$$= \phi + i\psi$$

$\therefore$ streamfunction, $\psi = U(y - ae^{-ky}\sin kx)$.

for $y \sim a$ and $ka \ll 1$ $\quad e^{-ky} \to 1$

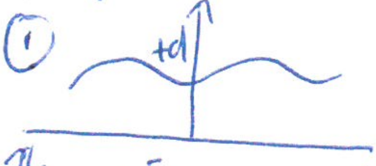
$\therefore$ streamline $\psi = 0$ lies on $\underline{y = a\sin kx}$.

(b)



$\left\{\begin{array}{l}\text{ka small}\\ \text{kd } \underline{\text{NOT}} \text{ small}\end{array}\right.$

superpose $\underline{\text{two}}$ streamfunctions $\psi_1 + \psi_2 = \psi$.

① 

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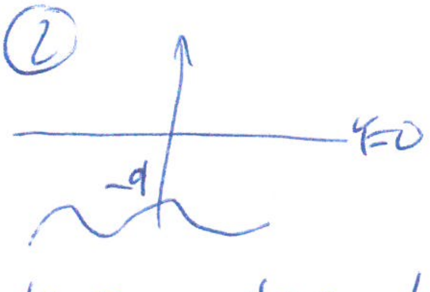
② 

write $\tilde{a}$ as similar to part (a).

$$\left\{\begin{array}{l}\psi_1 = U[(y-d) - \tilde{a}e^{+k(y-d)}\sin kx]\\ \psi_2 = U[(y+d) - \tilde{a}e^{-k(y+d)}\sin kx]\end{array}\right. \text{ so } y=0 \text{ is streamline}$$

$$\therefore \psi = Uy - U\tilde{a}\sin kx\left(e^{k(y-d)} - e^{-k(y+d)}\right) \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$= Uy - 2U\tilde{a}e^{-kd}\sin kx \sinh ky$$

$\tilde{a}$ will be adjusted later...

Check: on $Y=0$, $\Psi=0$ ✓.

The maximum velocity in this flow will be $(u,0)$ at the points where $\sin kx = +1$ or -1 i.e. at the "peaks" so can simplify algebra:

$$u = \frac{\partial \Psi}{\partial Y} = U - 2U\tilde{a}e^{-kd}\underbrace{\sin kx}_{\substack{+1 \text{ max} \\ -1 \text{ min}}} \cdot k\cosh kY$$

$$\therefore \underline{U_{max} - U_{min} = 4U\tilde{a}e^{-kd}k\cosh kd}$$

Now, evaluate "$\tilde{a}$" such that the superposed flows are streamlines over each corrugated surface:

$$\Psi_{(Y=-d+a\sin kx)} = U(-d+a\sin kx) - 2U\tilde{a}e^{-kd}\sinh(-kd + \underbrace{ka\sin kx}_{\ll 1}) \times \sin kx$$

$$= -Ud + U\left[\underline{a\sin kx} - 2\tilde{a}e^{-kd}\sinh(-kd)\underline{\sin kx}\right]$$

$\Psi=0$ on $Y=0$ $\therefore$ $\Psi = -Ud$ on lower wall (vol. flow $Ud = \Delta\Psi$)

$\therefore [\cdots] = 0$ $\therefore$ $a = 2\tilde{a}e^{-kd}\sinh(-kd)$

$$\therefore U_{max} - U_{min} = \frac{4U.a.\cancel{e^{-kd}}.k\cosh kd}{\cancel{e^{-kd}}\sinh(-kd)} = \underline{4Ukd\coth(kd)}$$

4. (a) The incidence & camber solutions for a thin airfoil vortex sheet are:

$$\gamma(\phi) = -\pi U\left[(2\alpha+g_0)\frac{1-\cos\phi}{\sin\phi} + \sum_{n=1}^{\infty} g_n \sin(n\phi)\right]$$

$$\frac{x}{c} = \frac{1}{2}(1+\cos\phi)$$

The $(1-\cos\phi)/\sin\phi$ term tends to infinity at the LE ($x=0$, $\phi\to\pi$) and this implies a strong adverse pressure gradient on the suction side. This in turn implies a very distressed boundary layer and hence high drag or even stall. However, if $(2\alpha+g_0)=0$ then this singular term disappears and so we would expect minimum drag at AoA $\alpha = -\frac{1}{2}g_0$.

(b) Camber line $\frac{y_c}{c} = Y\left[\left(\frac{x}{c}\right)^3 - 3\left(\frac{x}{c}\right)^2 + 2\left(\frac{x}{c}\right)\right]$

$$-2\frac{dy_c}{dx} = -2\frac{d(y_c/c)}{d(x/c)} = -2Y\left[3\left(\frac{x}{c}\right)^2 - 6\left(\frac{x}{c}\right) + 2\right] \quad ; \frac{x}{c} = \frac{1}{2}(1+\cos\theta)$$

$$\therefore -2\frac{dy_c}{dx} = Y\left[-6\left(\frac{1+\cos\theta}{2}\right)^2 + 12\left(\frac{1+\cos\theta}{2}\right) - 4\right]$$

$\frac{6}{2}-4-\frac{3}{2}$

$$= Y\left[-\frac{3}{2}(1+2\cos\theta+\cos^2\theta) + 6(1+\cos\theta) - 4\right]$$

$6-3$

$$= Y\left[\frac{1}{2} + 3\cos\theta - \frac{3}{2}\left(\frac{\cos 2\theta+1}{2}\right)\right]$$

$\therefore -2\frac{dy_c}{dx} = \gamma\left[-\frac{1}{4} + 3\cos\theta - \frac{3}{4}\cos 2\theta\right]$

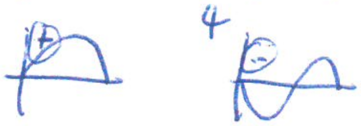
Hence, by inspection $g_0 = -\frac{1}{4}\gamma$, $g_1 = 3\gamma$, $g_2 = -\frac{3}{4}\gamma$

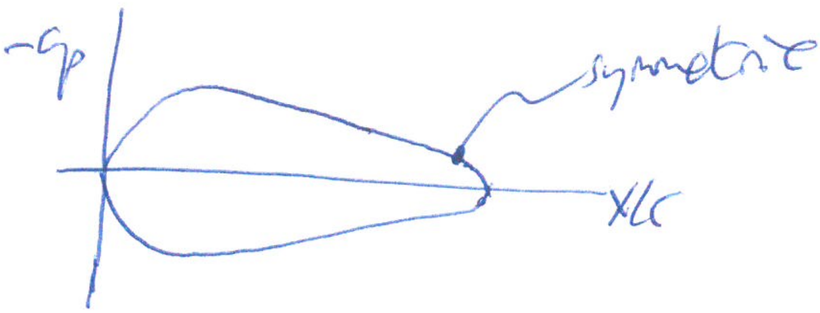
So:

(i) "design" incidence, $\underline{\alpha = -\frac{1}{2}g_0 = \frac{1}{8}\gamma}$

(ii) $\underline{-C_p = \mp\frac{\gamma}{U}} = \pm\gamma\left[3\sin\theta - \frac{3}{4}\sin 2\theta\right]$ since $2\alpha + g_0 = 0$

"vel. jump"

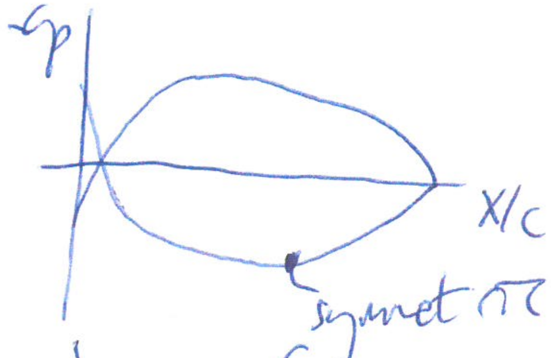




(iii) at zero incidence

$\alpha = 0$

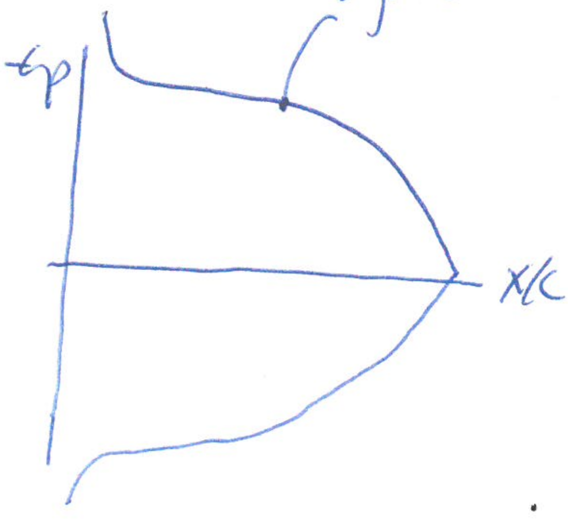
$\therefore -C_p = \pm\gamma\left[-\frac{1}{4}\frac{1-\cos\theta}{\sin\theta} + 3\sin\theta - \frac{3}{4}\sin 2\theta\right]$



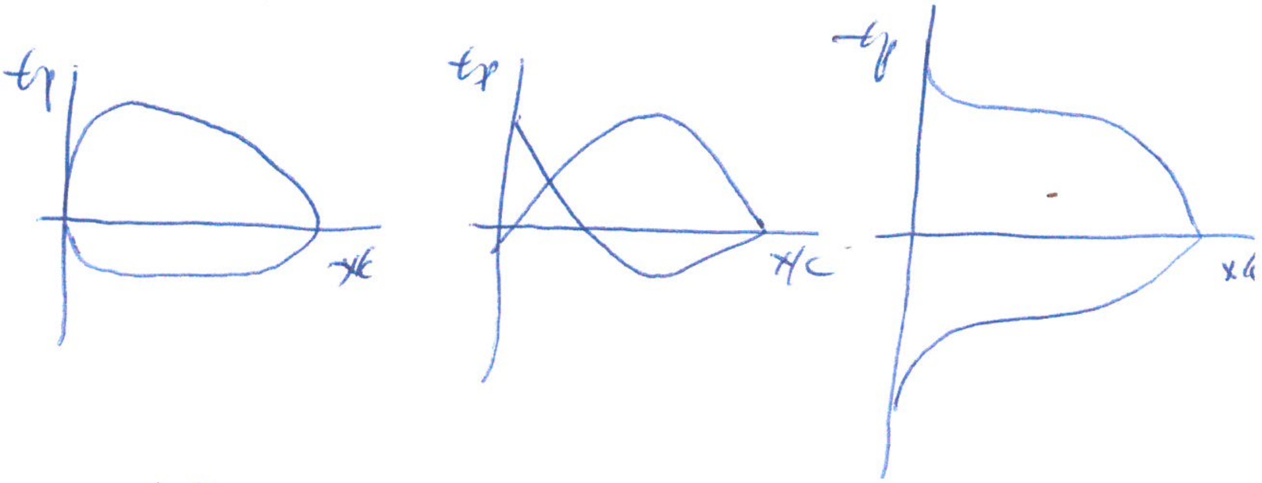
& at twice design AoA

$\alpha = -g_0$

$\therefore -C_p = \pm\gamma\left[+\frac{1}{4}\cdot\frac{1-\cos\theta}{\sin\theta} + 3\sin\theta - \frac{3}{4}\sin 2\theta\right]$

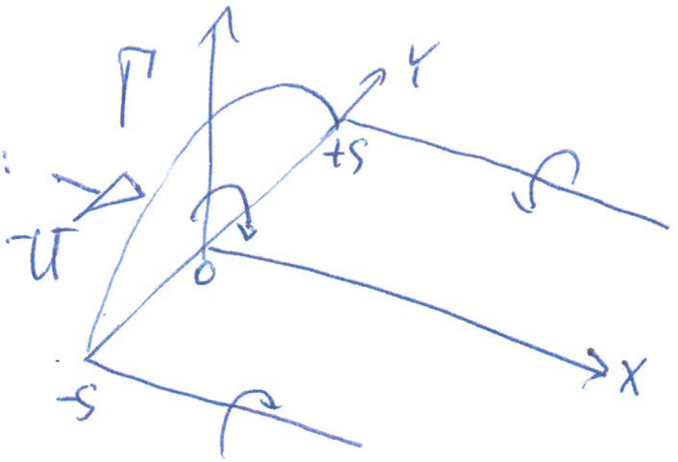


(c) The effect of thickness is to add an asymm=
=etric contribution, tending to move load
forward thus reducing adverse pressure grad=
=ients near the LE (thus improving b'layer flow
behaviour) and vice versa towards TE.



$\alpha =$ design   zero   twice-design

(5) Lifting line:



(a) Thin-airfoil theory for a 2D symmetrical section has:

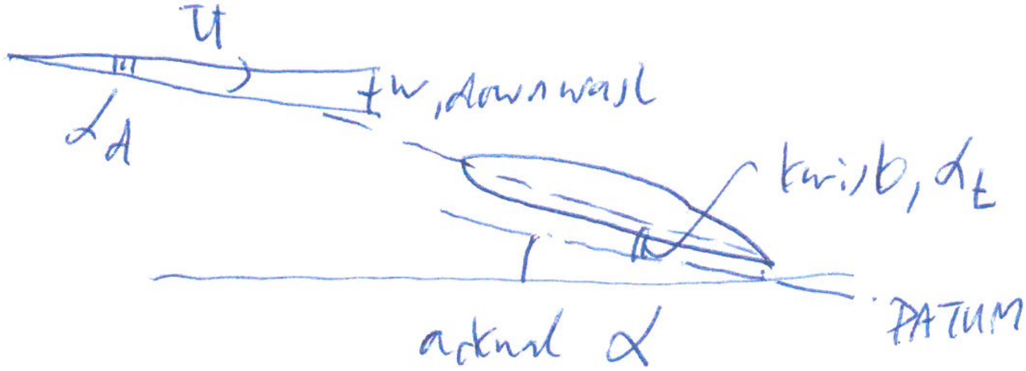
$$C_L = 2\pi\alpha$$

incidence

3D finite wing lifting line theory assumes this to hold at each individual section and that the "effective" angle of incidence is made up from the actual angle, the wing twist angle and the downwash angle,

$$C_L(y) = 2\pi\left[\alpha + \alpha_t(y) - \alpha_d(y)\right]$$

actual angle   twist angle   downwash angle.



(b) $\frac{\Gamma}{Us} = G_1(\sin\theta + a\sin 3\theta)$ where $y = -s\cos\theta$

DATA BOOK: $\alpha_d(y) = \frac{1}{4\pi U}\int_{-s}^{+s} \frac{d\Gamma}{dy'} \cdot \frac{dy'}{y - y'}$

$$= \frac{1}{4\pi U}\int \frac{d\Gamma}{d\theta} \cdot \cancel{\frac{d\theta}{dy}} \cdot \frac{1}{s\cos\phi - s\cos\theta} \cdot \cancel{\frac{dy}{d\theta}} \cdot d\theta$$

$$\alpha_d(\phi) = \frac{G_1 \cancel{Us}}{4\pi \cancel{Us}}\int_0^{\pi} \frac{\cos\theta + 3a\cos 3\theta}{\cos\theta - \cos\phi}\, d\theta$$

Glauert Integral:

$\int_0^{\pi} \frac{\cos n\theta \, d\theta}{\cos\theta - \cos\phi} = \frac{\pi \sin n\phi}{\sin\phi}$

DATA BOOK.

$$= \frac{G_1}{4\pi}\left[\int_0^{\pi}\left(\frac{\cos\theta}{\cos\theta - \cos\phi} + \frac{3a \cdot \cos 3\theta}{\cos\theta - \cos\phi}\right) d\theta\right]$$

$$= \frac{G_1}{4\pi}\left[\pi + 3a \cdot \frac{\sin 3\phi}{\sin\phi}\pi\right]$$

$$\therefore \underline{\alpha_d(\phi) = \tfrac{1}{4} G_1 \left(1 + 3a \frac{\sin 3\phi}{\sin\phi}\right)}$$

(c) For spanwise uniform lift coefficient, C_{LA}, the lifting line equation shows that

$$\alpha_T(y) - \alpha_d(y) = \text{constant} = -\alpha_d(0)$$

$\uparrow$ twist datum at $y = 0$

$$-\alpha_d(0) \qquad \phi = 0 \qquad +\alpha_d(y_s)$$

$$\therefore \alpha_t(y) = \frac{1}{4}G_1\left[-\left(\cancel{1}+3a\frac{\sin 3\theta}{\sin\theta}\right)+\left(\cancel{1}+3a\frac{\sin 3\phi}{\sin\phi}\right)\right]$$

$$\underset{x\to 0}{Lt}\ \frac{\sin 3x}{\sin x} = \underset{x\to 0}{Lt}\left(\frac{3x}{x}\right) = 3$$

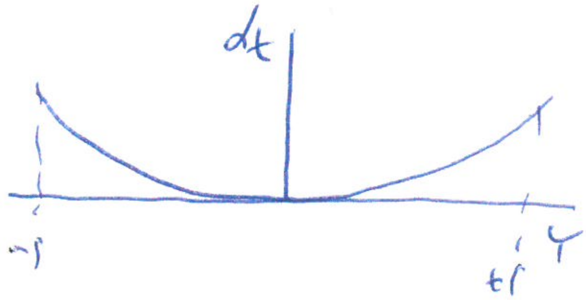
$$\therefore \alpha_t(y) = \frac{3a}{4}G_1\left(3+\frac{\sin 3\phi}{\sin\phi}\right)$$

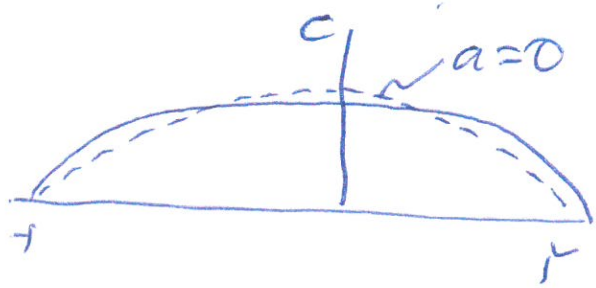
$G_1 U_s(\sin\theta + a\cos 3\theta)$

The local lift coefficient $C_l = \dfrac{\rho U \Gamma}{\frac{1}{2}\rho U^2 c} = \dfrac{-2\Gamma}{U c}$

constant C_{Ld} chord

$$\therefore \text{chord, } c = \frac{2G_1 s}{C_{Ld}}(\sin\phi + a\sin 3\phi)$$





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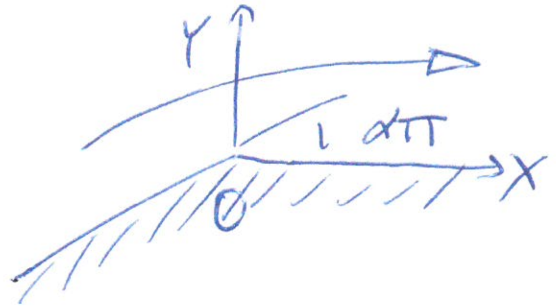


(a) Choose as complex potential $F(z) = Bz^n$

$$z = r\underbrace{e^{i\theta}}_{\cos\theta + i\sin\theta} \therefore F(z) = Br^n\cos n\theta + i\underbrace{Br^n\sin n\theta}_{\psi}$$

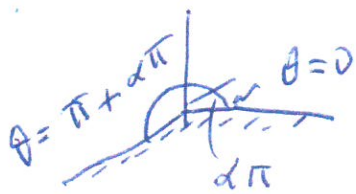
$$\therefore u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = Br^{n-1}n\cos n\theta$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = Br^{n-1}n\sin n\theta$$

$$U = \sqrt{u_r^2 + u_\theta^2} = nBr^{n-1}\underbrace{\sqrt{\cos^2 + \sin^2}}_{=1}$$

$$\therefore U = nBr^{n-1}$$

Now, for the wall to be a streamline, $\psi = 0$



$$\begin{cases} n\theta = 0 \\ n\theta = n(\pi + \alpha\pi) = \pi \end{cases} \quad \therefore n(1+\alpha) = 1$$

So for power-law type $U = kx^m$ $\quad m = n-1$

$$\therefore m = -\frac{\alpha}{1+\alpha}$$

(b) Similarity solution: $\psi(x,y) = F(x)f(\eta)$; $\eta = \frac{y}{g(x)}$

$$u = \partial\psi/\partial y = F(x)f'(\eta)/g(x) \to U(x) \text{ as } y \to \infty$$

as $f'(\infty) \to 1 \therefore F(x) = g(x)U(x) \quad \therefore \underline{\psi = U(x)g(x)f(\eta)}$

(c) B'layer equation (DATA BOOK):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

τ laminar $\mu\frac{\partial^2 u}{\partial y^2}$.

$$u = \frac{\partial \psi}{\partial y} = Ug f'/g = Uf'$$

$$v = -\frac{\partial \psi}{\partial x} = -U'gf - Ug'f - Ugf'\overbrace{\left(-\frac{y}{g^2}g'\right)}^{\partial\eta/\partial x}$$

$$= -U'gf - Ug'f + Uyf'g'/g.$$

$$\frac{\partial u}{\partial x} = U'f' - Uyg'f''/g^2$$

$$\frac{\partial u}{\partial y} = Uf''/g \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = Uf'''/g^2$$

(d) Substitute into the b'layer equations:

$$Uf'\left(U'f' - \frac{Uyg'f''}{g^2}\right) + \left(-U'gf - Ug'f + \frac{Uyf'g'}{g}\right)\frac{Uf''}{g}$$

$$= -\frac{1}{\rho}\frac{d}{dx}\overbrace{\left(p_0 - \frac{1}{2}\rho U^2\right)}^{p}\, + \nu Uf'''/g^2$$

$$\therefore UU'f'^2 - \cancel{U^2\frac{y}{g^2}g'f'f''} - UU'ff'' - U^2\frac{g'}{g}ff'' + \cancel{U^2\frac{y}{g^2}g'f'f''}$$

$$= UU' + \nu Uf'''/g^2$$

$$\therefore \cancel{U}U'f'^2 - \left(\cancel{U}U' + U^{\cancel{2}}\frac{g'}{g}\right)ff'' = \cancel{U}U' + \nu\cancel{U}f'''/g^2$$

$$\therefore \; f'^2 + \left(1 + \frac{U g'}{U' g}\right) f f'' = 1 + \frac{\nu}{g^2 U'} \cdot f''' \quad !$$

(e) A similarity solution exists if the terms with x-dependence become constant,

i.e. $\left(\frac{\nu}{g^2 U'}\right)$ = constant & $U' = mKx^{m-1}$ (from (a))

$$\therefore \; g^2 \sim x^{1-m} \quad \therefore \; g = c\, x^{\left(\frac{1-m}{2}\right)}$$

c constant.

So: $\left(1 + \frac{U g'}{U' g}\right) = 1 + \frac{Kx^m \cancel{c} \cdot \frac{1-m}{2} \cdot x^{-\frac{(1+m)}{2}}}{Km\, x^{m-1} \cancel{c}\, x^{\left(\frac{1-m}{2}\right)}}$

$$= 1 + \frac{1-m}{2m} \cdot x^{\cancel{m}+\cancel{1}-\cancel{m}-\frac{\cancel{1}+\cancel{m}}{2}+\frac{\cancel{m}-\cancel{1}}{2}=0} \quad !$$

$x^0 = 1$

recall, $m = \frac{-\alpha}{1+\alpha}$

$$= \frac{m+1}{2m} = \frac{1 - \frac{\alpha}{1+\alpha}}{\frac{-2\alpha}{1+\alpha}} = \frac{\frac{1}{\cancel{1+\alpha}}}{\frac{-2\alpha}{\cancel{1+\alpha}}}$$

$\therefore \left(1 + \frac{U g'}{U' g}\right) = -\frac{1}{2\alpha}$ choose some constant $\left(\frac{\nu}{g^2 U'}\right) = -\frac{1}{2\alpha}$

Hence: $\frac{1}{2\alpha} f''' + f'^2 - \frac{1}{2\alpha} f f'' = 1$

i.e. $\underline{f''' - f f'' + 2\alpha(f'^2 - 1) = 0}$

(f) For large turning angle the flow is likely to separate and so the solution invalid.

(7) Consider the b'layer eq^{ns}: [DATA BOOK]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \; ; \; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

(a) With external flow $U(x)$, $-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx}$

$$\therefore U\frac{dU}{dx} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\therefore (U-u)\frac{dU}{dx} + u\frac{dU}{dx} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\therefore (U-u)\frac{dU}{dx} + \frac{\partial uU}{\partial x} - U\frac{\partial u}{\partial x} - \frac{\partial u^2}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial vU}{\partial y} - U\frac{\partial v}{\partial y} - v\frac{\partial U}{\partial y}$$

$$-v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$-\frac{\partial}{\partial y}(uv) + u\frac{\partial v}{\partial y}$$

$$\therefore (U-u)\frac{\partial U}{\partial x} + \frac{\partial}{\partial x}(uU - u^2) + \frac{\partial}{\partial y}(vU - vu)$$

$$\underbrace{-U\frac{\partial u}{\partial x} - U\frac{\partial v}{\partial y}}_{-U\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0} + \underbrace{u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y}}_{u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0} = -\frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\therefore \frac{\partial}{\partial x}\left[u(U-u)\right] + \frac{\partial}{\partial y}\left[v(U-u)\right] + (U-u)\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

(b) Hence: $\frac{d}{dx}\int_0^\infty u(U-u)\,dy + \cancel{\int_0^\infty \frac{\partial}{\partial y}\left[v(U-u)\right]dy} + \frac{dU}{dx}\int_0^\infty (U-u)\,dy$

$V(0)=0$
$v(\infty)=0$

$$= -\frac{1}{\rho}\left[\tau\right]_0^\infty$$

$$\therefore U^2\frac{d}{dx}\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy + U\frac{dU}{dx}\int_0^\infty \left(1-\frac{u}{U}\right)dy = \tau_w/\rho .$$

$f(x)$!!!

$$\therefore \frac{d}{dx}(U^2\theta) + U\frac{dU}{dx}\delta^* = \tau_w/\rho$$

$$\therefore U^2\frac{d\theta}{dx} + 2\theta U\frac{dU}{dx} + U\frac{dU}{dx}\delta^* = \tau_w/\rho$$

$$\therefore \frac{d\theta}{dx} + \frac{1}{U}\frac{dU}{dx}(2+\underset{=\delta^*/\theta}{H})\theta = \frac{C_f}{2}$$

$\frac{\tau_w}{\frac{1}{2}\rho U^2}$

(c) With $U = \text{const.}$ and $u/U = \sin\left(\frac{\pi y}{2\delta}\right)$ then:

$$\theta = \int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right)\left(1-\sin\frac{\pi y}{2\delta}\right)dy = \frac{4-\pi}{2\pi}\delta$$

0.023

-0.142 / 6.283

$$\tau_w = \mu\frac{du}{dy}\Big|_{y=0} = \mu\frac{\pi U}{2\delta}\cos\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0} = \mu\frac{\pi U}{2\delta}.$$

$$\therefore \frac{d\theta}{dx} = \overbrace{\mu\frac{\pi U}{2\delta}}^{\tau_w}\cdot\frac{1}{\rho U^2} = \frac{\nu\pi}{2\delta U} = \frac{\nu\pi}{2\theta U}\cdot\frac{2\pi}{4-\pi}$$

$$\therefore \theta\frac{d\theta}{dx} = \frac{\nu}{U}\cdot\frac{\pi^2}{4-\pi} = \frac{1}{2}\frac{d}{dx}(\theta^2)$$

So, with $\theta = 0$ @ $x = 0$: $\quad \theta = \sqrt{\underbrace{\frac{2\pi^2}{4-\pi}}_{23.30}\left(\frac{\nu x}{U}\right)}$

$$\therefore \frac{\theta}{x} = 4.83\Big/\sqrt{\frac{Ux}{\nu}}.$$

(d) Comparing the sine-wave profile to the parabolic $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \ldots$

$$@ y=0 \quad \underbrace{u\frac{\partial u}{\partial y}}_{0} + \underbrace{v\frac{\partial u}{\partial x}}_{0} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{0} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$U = \text{constant}$

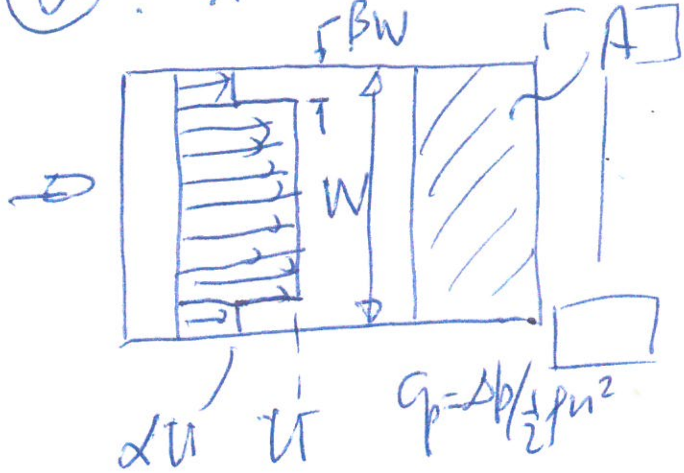
Leaving $\frac{\partial^2 u}{\partial y^2} = 0$.

for $u/U = \sin\left(\frac{\pi y}{2\delta}\right)$, $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = 0$

but $u/U = 2(y/\delta) - (y/\delta)^2$, $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{-2U}{\delta^2} \neq 0$!

which does not satisfy the "compatibility condition" — hence the sine-profile is more physical than the parabolic.

(8). A race car has an underbody diffuser width area A and pressure recovery coefficient C_p. It is fed by flow from up-stream.



(a) If the flow arrives with the non-uniformity in the sketch (from wheel wakes for example) then the downforce from the diffuser is:

$$L = \underbrace{(1-2\beta) A C_p \tfrac{1}{2}\rho U^2}_{\text{free-stream}} + \underbrace{2\beta A C_p \tfrac{1}{2}\rho (\alpha U)^2}_{\text{wakes}}$$

$$= \tfrac{1}{2}\rho U^2 C_p A\left[1-2\beta + 2\beta\alpha^2\right] = \tfrac{1}{2}\rho U^2 C_p A\left[1-2\beta(1-\alpha^2)\right]$$

(b) If the flow mixes out to uniform at diffuser entry (and no flow enters around the floor) then the diffuser sees flow $\bar{U}W = 2\beta W\alpha U + (1-2\beta)WU$

$$\therefore \cancel{W}\bar{U} = \cancel{W}U(1-2\beta+2\beta\alpha)$$

$$\therefore \bar{U} = U(1-2\beta(1-\alpha))$$

(c) $\therefore$ downforce now is $L_{MO} = \frac{1}{2}\rho U^2 C_p A\,(1-2\beta(1-\alpha))^2$

$$\therefore L_{MO}/L = \frac{[1-2\beta(1-\alpha)]^2}{[1-2\beta(1-\alpha)(1+\alpha)]}$$

Hence

$\frac{L_{NO}}{L}$	α	β
1	1	0
0.989	0.75	0.1
0.952	0.50	0.1
0.914	0.50	0.2

So there is significant loss of diffuser downforce if the onset flow non-uniformities are mixed out. This is because mass flow is linear in "u" but pressure is quadratic – "u²" – and the mean of the square is not equal to the square of the mean!

(d) To maintain the flow un-mixed vortex generators can be used to create a sort of barrier to mixing between the clean core flow and the wakes.