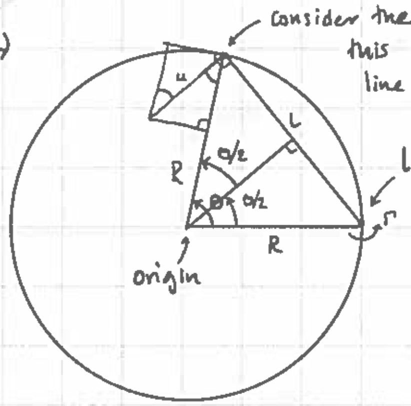
1. (a)



By geometric arguments:

$L = 2R\sin(\theta/2)$ distance from vortex

$u = \dfrac{\Gamma}{2\pi L}$ magnitude of velocity induced at (r,θ)

$\left.\begin{aligned} u_r &= -u\cos(\theta/2) \\ u_\theta &= u\sin(\theta/2) \end{aligned}\right\}$ components of u in the r and θ directions.

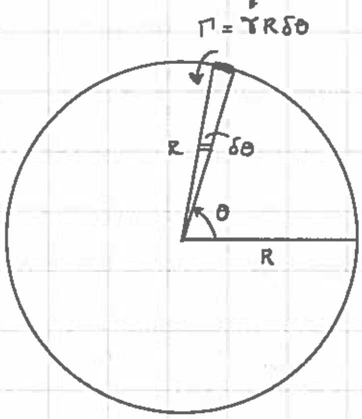
$$\Rightarrow u_r = \frac{-\Gamma}{4\pi R}\frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{-\Gamma}{4\pi R}\cot(\theta/2)$$

$$u_\theta = \frac{\Gamma}{4\pi R}$$

(b)

$$\Gamma = \int_0^{2\pi} \gamma R\, d\theta = 2\pi R\gamma$$

circulation per unit length: $2\pi R\gamma = \Gamma \rightarrow \gamma = \Gamma/2\pi R$



From (a) the velocity induced at $(R,0)$ by a vortex of strength Γ at (R,θ) is $u_r = \dfrac{\Gamma}{4\pi R}\cot(\theta/2)$

$$u_\theta = \frac{\Gamma}{4\pi R}$$

Substituting $\Gamma = \gamma R\delta\theta$ and integrating from $\theta=0$ to 2π:

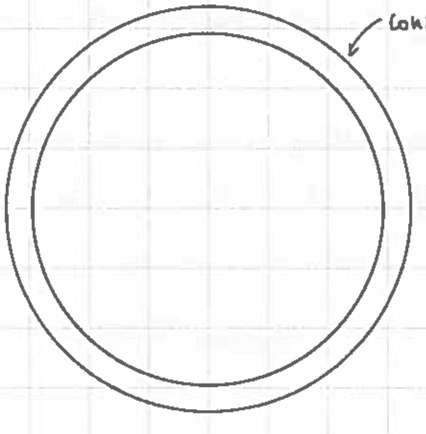
$$u_r = \frac{\gamma}{4\pi}\int_0^{2\pi}\cot(\theta/2)\,d\theta = 0$$ (Obvious from sketch of $\cot(\theta/2)$ or via symmetry)

$$u_\theta = \frac{\gamma}{4\pi}\int_0^{2\pi} d\theta = \frac{\gamma}{2}$$

so the sheet itself has velocity $u_r = 0$, $u_\theta = \dfrac{\gamma}{2} = \dfrac{\Gamma}{4\pi R}$

The vortex sheet rotates at speed $u_\theta = \Gamma/4\pi R$ around the origin. Because the sheet itself is uniform, the flow is steady (i.e. the flow does not change in time).

(c)



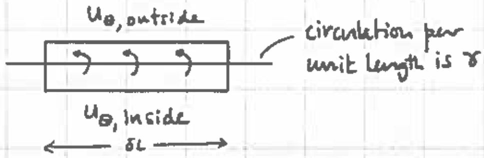
consider the circulation enclosed within the green contour, which lies just outside the vortex sheet. $\oint \underset{\sim}{u}\cdot dl = \Gamma$ because the vortex sheet has total circulation Γ from (b).

The flow is axisymmetric so u_θ is uniform around the green contour. Therefore $u_\theta = \dfrac{\Gamma}{2\pi R}$.

The radial velocity is zero. (This is obvious, but could be confirmed by considering $\nabla\cdot\underset{\sim}{u} = 0$ for a flow in which u_θ is uniform.)

(d) Calculate the velocity jump across the sheet using Stokes' theorem $\int \omega \cdot ds = \oint_c u \cdot dl$

Consider a small element of the sheet:



The circulation enclosed within the box is $\int \omega \cdot ds = \gamma \delta l$

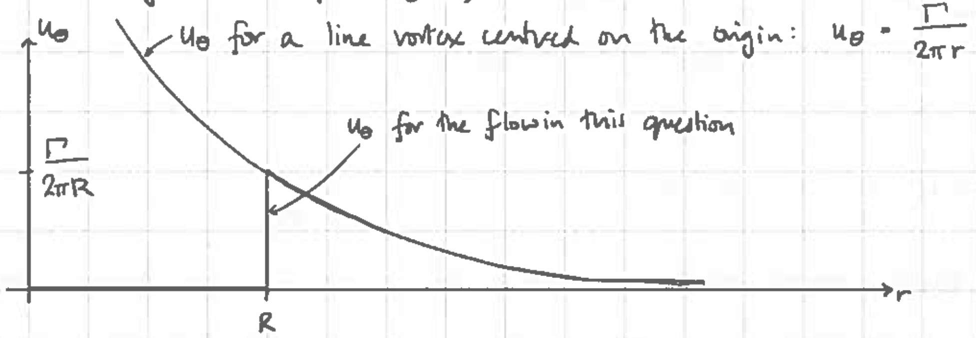
By Stokes' theorem, $\int \omega \cdot ds = \oint u \cdot dl = (u_{\theta, outside} - u_{\theta, inside})\, \delta l = \gamma \delta l$

$$\Rightarrow (u_{\theta, outside} - u_{\theta, inside}) = \gamma = \frac{\Gamma}{2\pi R}$$

The velocity jump across the vortex sheet is $\Delta u_\theta = \dfrac{\Gamma}{2\pi R}$.

Another method is to consider the circulation enclosed within a closed contour just inside the vortex sheet. This is zero, so $u_{\theta, inside} = 0$ and, from (c), the velocity jump is $\Gamma / 2\pi R$.

(e) From (c) and (d) we can infer that $u_\theta = 0$ for $r < R$. From (c), we see that for $r > R$, the velocity is identical to that of a line vortex with strength Γ centred on the origin. Therefore $u_\theta(r)$ is:



(f) Advantages: This is a physically accurate way to add the influence of rotation onto a cylinder. The flow inside the cylinder is unaffected by the vortex sheet, but the flow outside has circulation Γ added to it.

Disadvantages: It is considerably more complicated than simply adding a line vortex at the centre of the cylinder, which has exactly the same effect outside the cylinder. Therefore, if one is not concerned about the flow inside the cylinder there is no advantage in using the vortex sheet model.

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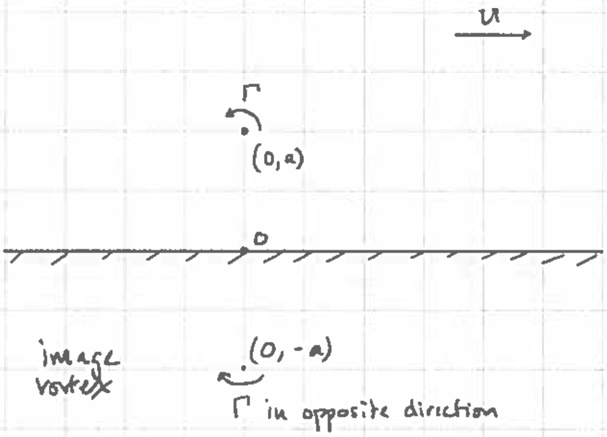
Jan 2019.

2. (a)



Let us define U to be positive in the positive x-direction. The velocity of the vortex at $(0,a)$ is therefore:

$$U + \frac{\Gamma}{4\pi a}$$

For this to be zero, $U = -\dfrac{\Gamma}{4\pi a}$

(n.b. the flow is from right to left)

The complex potential is: $F(z) = -\dfrac{\Gamma z}{4\pi a} - \dfrac{i\Gamma}{2\pi}\ln(z-ia) + \dfrac{i\Gamma}{2\pi}\ln(z+ia)$

(b) The stagnation points lie where $dF/dz = 0$ and, by inspection have $\mathrm{Im}(z)=0$

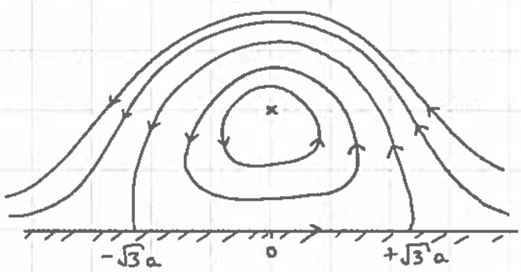
$$\frac{dF}{dz} = -\frac{\Gamma}{4\pi a} - \frac{i\Gamma}{2\pi}\frac{1}{(z-ia)} + \frac{i\Gamma}{2\pi}\frac{1}{(z+ia)} = 0$$

$$\Rightarrow \quad \frac{i}{z+ia} - \frac{i}{z-ia} = \frac{1}{2a}$$

$$\Rightarrow \quad i\,\frac{(z-ia)-(z+ia)}{z^2+a^2} = \frac{1}{2a}$$

$$\Rightarrow \quad \frac{2a}{z^2+a^2} = \frac{1}{2a}$$

$$\Rightarrow \quad 4a^2 = z^2+a^2 \quad \Rightarrow \quad z = \pm\sqrt{3}\,a$$

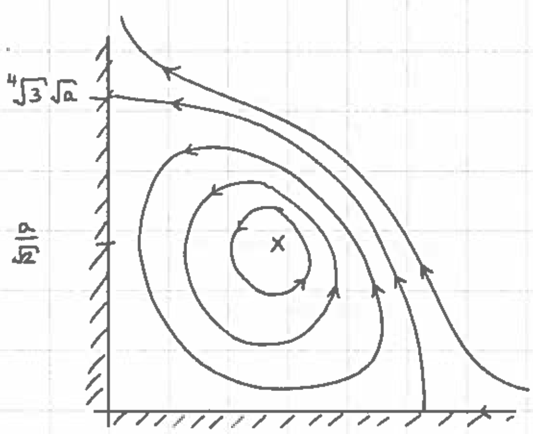


(c) For the boundaries of this flow to make an angle of 90° at the origin, the mapping required is $\zeta = z^{1/2}$. In the ζ-plane, the complex potential is found by setting $z=\zeta^2$:

$$F(\zeta) = -\frac{\Gamma\zeta^2}{4\pi a} - \frac{i\Gamma}{2\pi}\ln(\zeta^2-ia) + \frac{i\Gamma}{2\pi}\ln(\zeta^2+ia)$$

The stagnation points are at $\pm\sqrt{\pm\sqrt{3}\,a}$. For the 90° corner shown in Fig 1 they are at $\sqrt[4]{3}\sqrt{a}$ $\quad \sqrt[4]{3}\,i\sqrt{a}$.

The vortex moves to $\dfrac{(1+i)}{\sqrt{2}}a$



(d) The flow is symmetric around the line $\zeta_i = \zeta_r$ so let us consider only points on the ζ_r axis. Along this axis, the velocity is horizontal, so $dF/d\zeta$ is real. The lowest pressure point will be that at which $|dF/d\zeta|$ is highest and $d^2F/d\zeta^2$ is zero.

$$\frac{dF}{d\zeta} = \frac{dz}{d\zeta}\frac{dF}{dz} = 2\zeta\frac{\Gamma}{2\pi}\left\{-\frac{1}{2a} - \frac{i}{(z-ia)} + \frac{i}{(z+ia)}\right\} = \frac{\Gamma\zeta}{\pi}\left\{-\frac{1}{2a} + i\frac{(z-ia)-(z+ia)}{z^2+a^2}\right\}$$

$$\boxed{\frac{dz}{d\zeta} = 2\zeta}$$

$$= \frac{\Gamma z^{1/2}}{\pi}\left\{-\frac{1}{2a} + \frac{2a}{z^2+a^2}\right\}$$

This is true for complex z. But now let's set $z = x$ (real) to get $dF/d\zeta$ along the real z axis:

$$dF/d\zeta = \frac{\Gamma x^{1/2}}{\pi}\left\{\frac{2a}{x^2+a^2} - \frac{1}{2a}\right\}$$ along the real z axis, where $\zeta_r = z_r^{1/2}$

We need to find x for which $dF/d\zeta = 0$ along the real z axis.

$$\frac{d}{dx}\left\{\frac{\Gamma x^{1/2}}{\pi}\left(\frac{2a}{x^2+a^2} - \frac{1}{2a}\right)\right\} = \frac{\Gamma}{\pi}\frac{1}{2}x^{-1/2}\left(\frac{2a}{x^2+a^2} - \frac{1}{2a}\right) + \frac{\Gamma}{\pi}x^{1/2}\left(\frac{-4ax}{(x^2+a^2)^2}\right) = 0$$

$$\Rightarrow \frac{\Gamma}{\pi}\left(\frac{a}{x^2+a^2} - \frac{1}{4a} - \frac{4ax^2}{(x^2+a^2)^2}\right) = 0$$

$$\Rightarrow 4a^2(x^2+a^2) - (x^2+a^2)^2 - 16a^2x^2 = 0$$

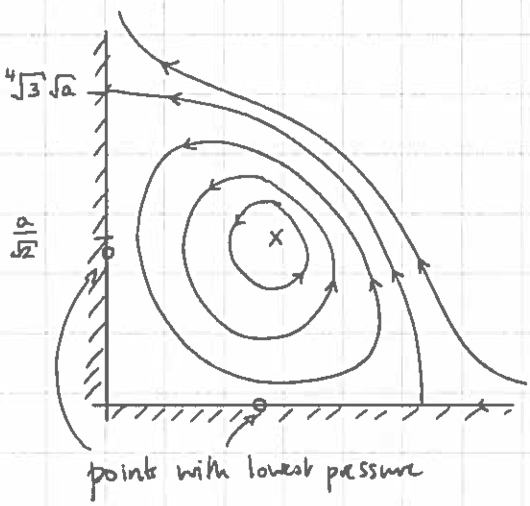
$$\Rightarrow 4a^2x^2 + 4a^4 - x^4 - 2x^2a^2 - a^4 - 16a^2x^2 = 0$$

$$\Rightarrow x^4 + 14a^2x^2 - 3a^4 = 0$$

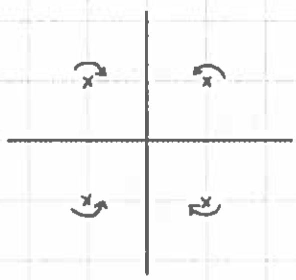
$$\Rightarrow 2x^2 = -14a^2 \pm (14^2a^4 + 12a^4)^{1/2}$$

$$\Rightarrow x^2 = 0.2111\,a^2$$

On the ζ_r axis, this point is at $\sqrt[4]{0.2111}\;a^{1/2} = 0.6778\,a^{1/2}$



(e) The image system is:



Yes, the vortices remain stationary. They must do because the flow in the z-plane was constructed such that they remain stationary. The mutually-induced velocities of each vortex are exactly balanced by the mean flow, U.

Matthew Juniper
Jan 2019.

3. (a) The fourth and fifth terms can be combined:

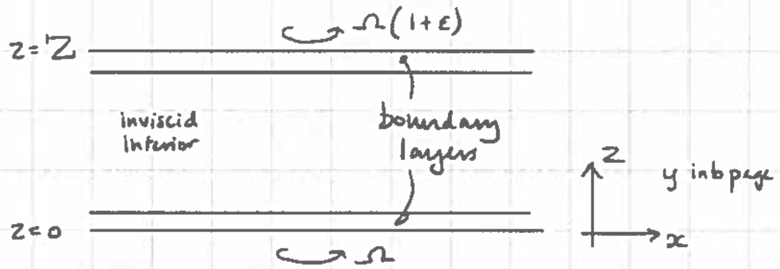
$$-\frac{1}{\rho}\nabla p - \Omega\times(\Omega\times x) = -\frac{1}{\rho}\nabla p + \nabla\left(\frac{(\Omega\times x)^2}{2}\right)$$
$$= -\frac{1}{\rho}\nabla\left(p - \frac{\rho}{2}(\Omega\times x)^2\right)$$
$$= -\frac{1}{\rho}\nabla P \text{ , where } P = p - \frac{\rho}{2}(\Omega\times x)^2$$

This term is the centrifugal force that appears in the rotating reference frame equations. It can be treated by simply adding a scalar to the pressure field.

(b) $\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}$ scales with U^2/L ; $\Omega\times u$ scales with ΩU ; their ratio is $\frac{U^2/L}{\Omega U} = \frac{U}{\Omega L}$

The inertial forces are much smaller than the Coriolis forces when $U \ll \Omega L$. (This will occur on large lengthscales, for example in weather systems.)

(c)



In the inviscid interior, $2\underset{\sim}{\Omega}\times\underset{\sim}{u}_I = -\frac{1}{\rho}\nabla P_I$ (with the viscous term removed)

$$2\underset{\sim}{\Omega}\times\underset{\sim}{u} = 2\begin{pmatrix}0\\0\\\Omega\end{pmatrix}\times\begin{pmatrix}u_I\\v_I\\w_I\end{pmatrix} = 2\begin{pmatrix}-v_I\Omega\\+u_I\Omega\\0\end{pmatrix} = -\frac{2}{\rho}\begin{pmatrix}\partial P_I/\partial x\\\partial P_I/\partial y\\\partial P_I/\partial z\end{pmatrix}$$

z-component $\Rightarrow \partial P_I/\partial z = 0$ ie. the pressure is uniform in the z direction throughout the inviscid region. It follows immediately from the x and y components that u_I and v_I do not vary in z.

Further, $\nabla\cdot\underset{\sim}{u}_I = 0 \Rightarrow \frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} + \frac{\partial w_I}{\partial z} = 0$

$$\Rightarrow \frac{1}{2\rho\Omega}\left(-\frac{\partial^2 P_I}{\partial x\,\partial y} + \frac{\partial^2 P_I}{\partial y\,\partial x}\right) + \frac{\partial w_I}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w_I}{\partial z} = 0$$

We see that $\underset{\sim}{u}_I$ does not vary in z. This shows that, in a flow in which $U/\Omega L \ll 1$, the flow is two-dimensional. Any local changes to the flow must propagate in the z-direction until the flow becomes uniform in the z-direction.

(d) Using the previous result and including the viscous terms gives:

$$\begin{pmatrix}0\\0\\\Omega\end{pmatrix} \times \begin{pmatrix}u\\v\\w\end{pmatrix} = 2\begin{pmatrix}-v\Omega\\+u\Omega\\0\end{pmatrix} = -\frac{1}{\rho}\begin{pmatrix}\partial P/\partial x\\\partial P/\partial y\\\partial P/\partial z\end{pmatrix} + \nu\begin{pmatrix}\nabla^2 u\\\nabla^2 v\\\nabla^2 w\end{pmatrix}$$

$$\nabla^2 = \left(\underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}_{\text{small compared with } \partial^2/\partial z^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\Rightarrow \quad -2v\Omega = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$2u\Omega = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$ within the boundary layers.

(e) In the inviscid interior, $\frac{\partial P_I}{\partial x} = 2\rho\Omega v_I$ and $\frac{\partial P_I}{\partial y} = -2\rho\Omega u_I$.

Assuming that $\partial P/\partial z$ is zero throughout the whole flow, including the boundary layer, we can substitute these values into the above equations:

$$-2\Omega(v - v_I) = \nu\frac{\partial^2 u}{\partial z^2} \quad ; \quad 2\Omega(u - u_I) = \nu\frac{\partial^2 v}{\partial z^2}$$

Now define $f = u - u_I + i(v - v_I)$ and note that $\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 u}{\partial z^2} + i\frac{\partial^2 v}{\partial z^2}$ because u_I and v_I do not depend on z.

$$\Rightarrow 2\Omega f = 2\Omega(u - u_I + i(v - v_I))$$

$$= \nu\frac{\partial^2 v}{\partial z^2} - i\nu\frac{\partial^2 u}{\partial z^2} = \frac{\nu}{i}\left(\frac{\partial^2 u}{\partial z^2} + i\frac{\partial^2 v}{\partial z^2}\right) =$$

$$\Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{2\Omega}{\nu} i f$$

Assume solutions of the form $f(x,y,z) = \hat{f}(x,y)e^{sz}$ $\Rightarrow$ $\frac{\partial^2 f}{\partial z^2} = s^2 f$

By inspection, $s^2 = \frac{2\Omega i}{\nu}$, so $s = \pm(1+i)\sqrt{\frac{\Omega}{\nu}}$

The general solution is $f = Ae^{(1+i)\sqrt{\frac{\Omega}{\nu}}z} + Be^{-(1+i)\sqrt{\frac{\Omega}{\nu}}z}$

The boundary layer thickness is of order $(\nu/\Omega)^{1/2}$

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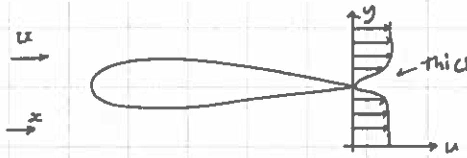
Jan 2019.

4.



thick wake in which $\underline{u}$ varies more rapidly in y than in x.

(a) Prandtl's boundary layer analysis applies here. We are told that $\underline{u}$ varies more rapidly in y than in x. We then examine the order of magnitude of the terms in the Navier-Stokes equations and consider the region in which inertia, pressure gradient, and viscous forces are all equally important. This gives rise to Prandtl's boundary layer equations, which we can quote from the datasheet:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (x\text{-momentum}) \;;\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{continuity})$$

Note that the free stream has uniform x-velocity, which means that $\partial p/\partial x = 0$, so

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \;;\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1a,\ 1b)$$

with boundary conditions $u \to U$ as $y \to \pm\infty$. (and $v \to 0$ as $y \to \pm\infty$)

(b) Sufficiently far downstream, $u \simeq U + u_1$, where $|u_1| \ll U$. Substitute into the momentum equation (1a):

$$(U + u_1)\frac{\partial u_1}{\partial x} + v\frac{\partial u_1}{\partial y} = \nu\frac{\partial^2 u_1}{\partial y^2}$$

neglecting high order terms leads to: $U\dfrac{\partial u_1}{\partial x} = \nu\dfrac{\partial^2 u_1}{\partial y^2}$ (2)

Integrate (2): $U\dfrac{\partial}{\partial x}\displaystyle\int_{-\infty}^{\infty} u_1\,dy = \nu\int_{-\infty}^{\infty}\frac{\partial^2 u_1}{\partial y^2}\,dy = \nu\left[\frac{\partial u_1}{\partial y}\right]_{-\infty}^{+\infty} = 0$ because $\dfrac{\partial u_1}{\partial y} \to 0$ at $\pm\infty$

$\Rightarrow \displaystyle\int_{-\infty}^{\infty} u_1\,dy$ does not vary in x. (3)

(c) Seek a similarity solution of the form $u_1 = F(x)f(\eta)$; $\eta = \dfrac{y}{g(x)}$: $\Rightarrow \left.\dfrac{\partial y}{\partial \eta}\right|_x = g(x)$

Start from (3): $\displaystyle\int_{-\infty}^{\infty} u_1\,dy = \underset{\text{constant}}{M}$

$$M = \int_{-\infty}^{\infty} F(x)f(\eta)\,dy = F(x)\int_{-\infty}^{\infty} g(x)f(\eta)\,d\eta = F(x)\,g(x)\int_{-\infty}^{\infty} f(\eta)\,d\eta$$

$$\Rightarrow F(x) = \frac{M}{g(x)\displaystyle\int_{-\infty}^{\infty} f(\eta)\,d\eta} \sim \frac{1}{g(x)}$$

(d) $U\frac{\partial u_1}{\partial x} = \nu \frac{\partial^2 u_1}{\partial y^2}$ where $u_1 = F(x) f(\eta)$ and $\eta = \frac{y}{g(x)}$ so $\frac{\partial \eta}{\partial x} = -\frac{dg}{dx}\frac{y}{g^2} = \frac{-g'\eta}{g}$

LHS: $\frac{\partial u_1}{\partial x} = F'f + F\frac{\partial}{\partial x}f(\eta) = F'f + Ff'\frac{\partial \eta}{\partial x} = F'f - \frac{Ff'g'\eta}{g}$

RHS: $\frac{\partial^2 u_1}{\partial y^2} = F\frac{\partial^2 f}{\partial y^2} = F\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial y}f'\right) = F\left(\left(\frac{\partial \eta}{\partial y}\right)^2 f'' + \underbrace{\frac{\partial^2 \eta}{\partial y^2}}_{=0} f'\right) = \frac{Ff''}{g^2}$

Substitute into (2) $\Rightarrow$ $U\left[F'f - \frac{Ff'g'\eta}{g}\right] = \nu \frac{Ff''}{g^2}$

$\Rightarrow$ $Ug^2F'f - UgFf'g'\eta = \nu F f''$; divide by νF

$\Rightarrow$ $\frac{F'}{F}\frac{Ug^2}{\nu}f - \frac{Ugg'}{\nu}\eta f' = f''$ (4)

For a similarity solution to exist, $\frac{F'}{F}\frac{Ug^2}{\nu}$ and $\frac{Ugg'}{\nu}$ must be constant

(e) Now, $(g^2)' = 2gg'$. If $\frac{Ugg'}{\nu} = 1$ then $(g^2)' = \frac{2\nu}{U}$ $\Rightarrow$ $g^2 = \frac{2\nu x}{U}$ $\Rightarrow$ $g = \left(\frac{2\nu x}{U}\right)^{1/2}$

From (c) we know that $F \sim \frac{1}{g} \sim \frac{A}{\sqrt{x}}$ where A is a constant

$\Rightarrow$ $\frac{F'}{F} = \frac{-\frac{1}{2}Ax^{-3/2}}{Ax^{-1/2}} = -\frac{1}{2x}$

Substitute all into (4) : $\frac{F'}{F}\frac{Ug^2}{\nu}f - \frac{Ugg'}{\nu}\eta f' = f''$

$\Rightarrow$ $\frac{-1}{2x}2xf - \eta f' = f''$

$\Rightarrow$ $f'' + f + \eta f' = 0$ (5)

(f) (5) can be written $f'' + (\eta f)' = 0$ or $(f' + \eta f)' = 0$

Integrate this once to obtain $f' + \eta f = \text{const.}$

$u_1 \to 0$ as $\eta \to \infty$ $f' + \eta f = 0$ [check with Jie Li]

$\Rightarrow$ $f'/f = -\eta$ $\Rightarrow$ $f = e^{-\eta^2/2}$ where $\eta = y/g(x)$, so $\eta^2/2 = \frac{y^2}{2g^2} = \frac{y^2 U}{4\nu x}$

$u_1 = F(x) f(\eta) = \frac{A}{\sqrt{x}} e^{-\frac{U}{4\nu x}y^2}$

5. (a) The boundary layer equation at the wall is: $\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx}$

(i) $m = \frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_w = -\frac{\theta^2}{U\nu}U\frac{dU}{dx} = -\frac{\theta^2}{\nu}\frac{dU}{dx}$

(ii) The momentum integral equation (datasheet) is: $\frac{d\theta}{dx} + \frac{\overbrace{H}^{\text{shape factor}}+2}{U}\theta\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} = \frac{c_f'}{2}$

multiply by $U\theta/\nu$ to give: $\frac{U\theta}{\nu}\frac{d\theta}{dx} + \frac{\cancel{U}\theta}{\nu}\frac{H+2}{\cancel{U}}\theta\frac{dU}{dx} = \frac{\cancel{U}\theta}{\underbrace{\nu\rho}_{\mu}}\frac{\tau_w}{U^{\cancel{2}}} = \frac{\theta\tau_w}{U\mu} = L$

$$\Rightarrow \frac{U\theta}{\nu}\frac{d\theta}{dx} = (H+2)m + L$$

$$\Rightarrow U\frac{d\theta^2}{dx} = 2\nu\left[(H+2)m + L\right]$$

(iii) Assume that $2\left[(H+2)m + L\right] = 0.45 + 6m$

$\Rightarrow U\frac{d\theta^2}{dx} = \nu(0.45 + 6m)$; now $\nu m = -\theta^2\frac{dU}{dx}$

$\Rightarrow U\frac{d\theta^2}{dx} = 0.45\nu - 6\theta^2\frac{dU}{dx}$ $\Rightarrow U\frac{d\theta^2}{dx} + 6\theta^2\frac{dU}{dx} = 0.45\nu$

now, $\frac{d}{dx}(U^6\theta^2) = U^5\left(6\theta^2\frac{dU}{dx} + U\frac{d\theta^2}{dx}\right) = 0.45\nu U^5$

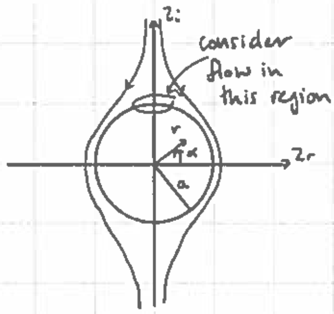
Integrate this equation from $x = 0$ to X:

$$\int_0^X \frac{d}{dx}(U^6\theta^2)\,dx = 0.45\nu\int_0^X U^5\,dx$$

$$\Rightarrow \left[U^6\theta^2\right]_0^X = U^6(X)\theta^2(X) - U^6(0)\theta^2(0) = 0.45\nu\int_0^X U(x)^5\,dx$$

$$\Rightarrow \theta^2(X) = \theta^2(0)\left(\frac{U(0)}{U(X)}\right)^6 + \frac{0.45\nu}{U(X)^6}\int_0^X U(x)^5\,dx \qquad [1]$$

(b) (i) $\phi = -U_\infty\left(r + \frac{a^2}{r}\right)\sin\alpha$



$u_r = \frac{\partial\phi}{\partial r} = -U_\infty\left(1 - \frac{a^2}{r^2}\right)\sin\alpha$; $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty\left(1 + \frac{a^2}{r^2}\right)\cos\alpha$

On $r = a$, $u_r = 0$ and $u_\alpha = -U_\infty 2\cos\alpha$

Further, on $r = a$, $x = a\cos\alpha$

$\Rightarrow u_\alpha = -2U_\infty\frac{x}{a}$ on the surface of the cylinder.

(ii) For $x \ll a$, the leading order approximation of the velocity around $x=0$, $y=a$ is:

$u = -U_\infty \sin\alpha \simeq 2U_\infty \frac{x}{a}$; $v = U_\infty \cos\alpha \simeq 0$ and now we take these as the free stream values for a boundary layer calculation.

Eq [1] becomes: $\theta^2(X) = \theta^2(0)\left(\frac{U(0)}{U(X)}\right)^6 + \frac{0.45\nu}{U(X)^6}\int_0^X U(x)^5\,dx$ where $U(0) = 0$; $U(x) = \frac{2U_\infty x}{a}$

$$\Rightarrow \theta^2(X) = 0.45\nu\left(\frac{a}{2U_\infty X}\right)^6 \int_0^X \left(\frac{2U_\infty}{a}\right)^5 x^5\,dx$$

$$= 0.45\nu\left(\frac{a}{2U_\infty}\right)\frac{1}{X^6}\left[\frac{x^6}{6}\right]_0^X$$

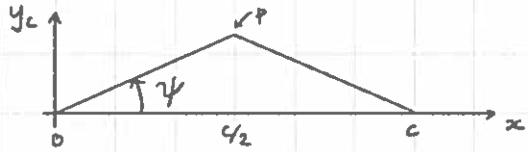
$$= \frac{0.45}{12}\frac{\nu a}{U_\infty} = 0.0375\frac{\nu a}{U_\infty}$$

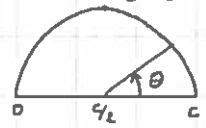
(iii) Notable features:

a) θ^2, the momentum thickness, is independent of x around the stagnation point.

b) $\left(\frac{\theta}{a}\right)^2 \propto \frac{\nu}{aU_\infty} = Re^{-1}$; the momentum thickness as a fraction of the body size, a, decreases proportional to $Re^{-1/2}$.

6



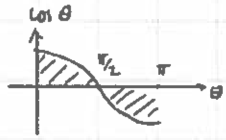


P ← $\theta = \pi/2$ at hinge, P.

$$x = \frac{c}{2}(1+\cos\theta)$$

$$y_c \simeq \psi x = \psi \frac{c}{2}(1+\cos\theta) \quad \text{for} \quad \theta \in (\pi/2, \pi]$$

$$y_c \simeq \psi c - \psi x = \psi \frac{c}{2}(1-\cos\theta) \quad \text{for} \quad \theta \in [0, \pi/2)$$



(a) From datasheet, $g_0 = \frac{1}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right)d\theta = \frac{1}{\pi}\left\{\int_0^{\pi/2} 2\psi \, d\theta + \int_{\pi/2}^{\pi} -2\psi \, d\theta\right\} = 0$

$$g_1 = \frac{2}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right)\cos\theta \, d\theta = \frac{2}{\pi}\left\{\int_0^{\pi/2} 2\psi \cos\theta \, d\theta + \int_{\pi/2}^{\pi} -2\psi\cos\theta \, d\theta\right\} = \frac{4\psi}{\pi}\left\{1+1\right\} = \frac{8\psi}{\pi}$$

$$g_2 = \frac{2}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right)\cos 2\theta \, d\theta = \frac{2}{\pi}\left\{\int_0^{\pi/2} 2\psi \cos 2\theta \, d\theta + \int_{\pi/2}^{\pi} -2\psi\cos 2\theta \, d\theta\right\} = 0$$

(b) From datasheet, $C_L = \pi\left(g_0 + \frac{g_1}{2}\right) = 4\psi$ for the camber solution. HB got $4\psi/\pi$ but I think that was an error.

The incidence solution is $C_{L_i} = 2\pi\alpha$

Therefore $C_L = 2\pi\alpha + 4\psi$

Are other g_n zero, or are we truncating at $n = 2$ here?

(c) From datasheet, $\gamma(L) = -U\left[g_0 \frac{1-\cos\phi}{\sin\phi} + \sum_1^\infty g_n \sin n\phi\right] = -U g_1 \sin\phi = -U\frac{8}{\pi}\psi\sin\phi$

Now $x = \frac{c}{2}(1+\cos\phi)$

$\Rightarrow \frac{2x}{c} - 1 = \cos\phi = \left(1-\sin^2\phi\right)^{1/2}$

$\Rightarrow \left(\frac{2x}{c}-1\right)^2 = 1-\sin^2\phi$

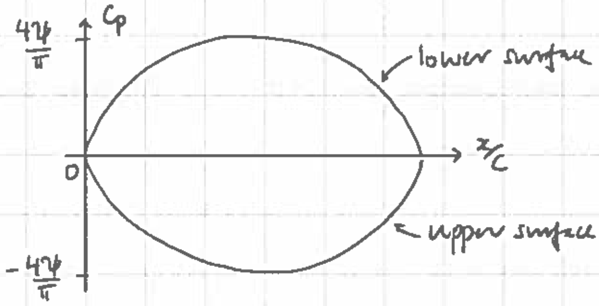
$\Rightarrow \sin\phi = \left(1-\left(\frac{2x}{c}-1\right)^2\right)^{1/2}$

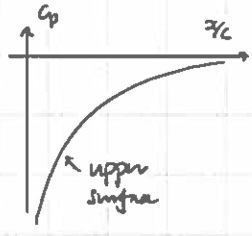
$1-(2\eta-1)^2 = 1-4\eta^2+4\eta-1$

$= 4\eta(1-\eta)$; $\eta = \frac{x}{c}$

$$\Rightarrow \gamma(L) = -U\frac{8}{\pi}\psi\left(1-\left(\frac{2x}{c}-1\right)^2\right)^{1/2}$$

The pressure coefficient is $C_p = \pm\frac{\gamma}{U} = \mp\frac{8\psi}{\pi}2\left(\eta(1-\eta)\right)^{1/2}$

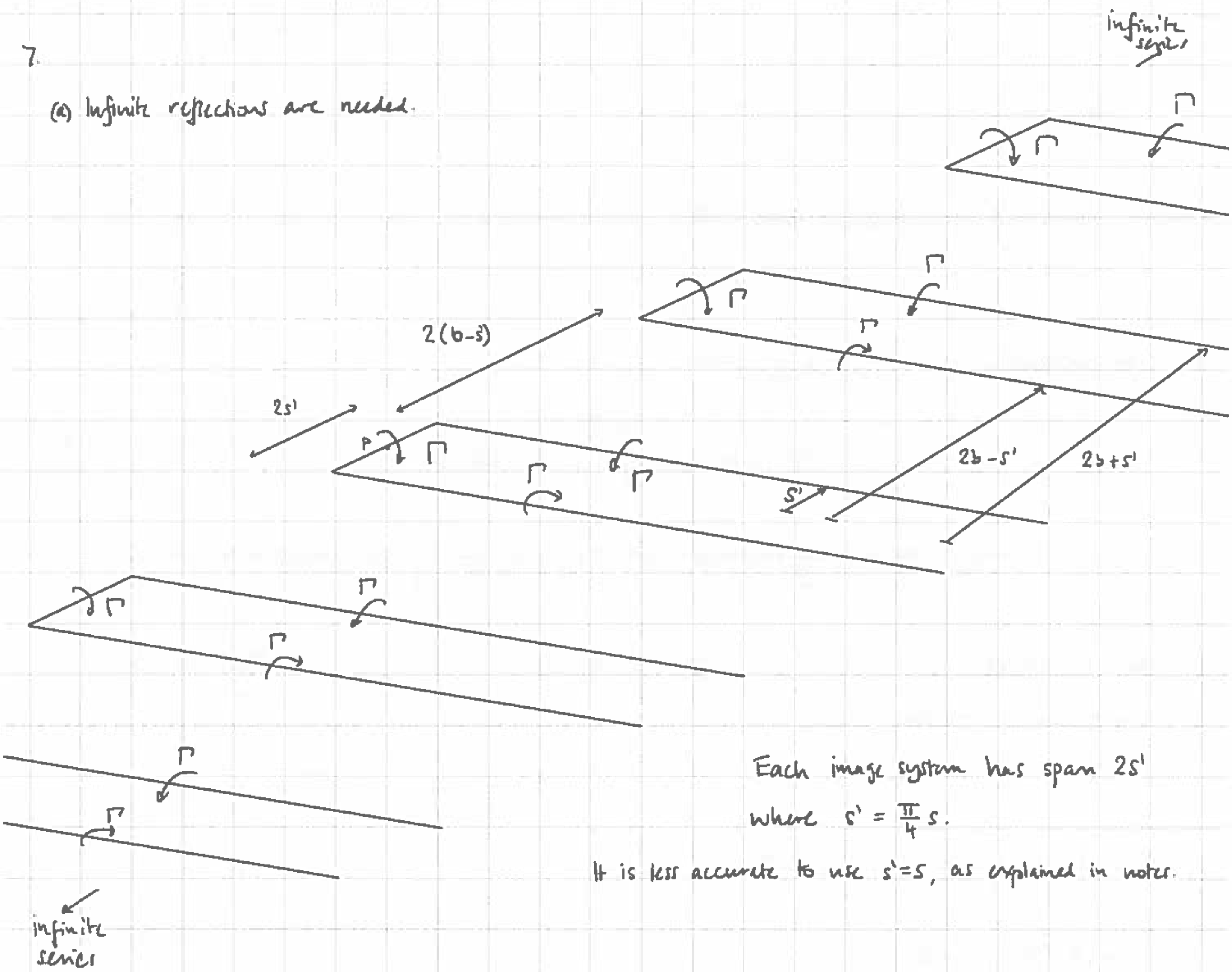




(d) From the flat plate solution in the notes, the incidence c_p distribution looks like this: Therefore the upper surface has a very sharp adverse pressure gradient at the leading edge. Compare this with $C_p(x)$ for the camber solution, shown above, which is much more benign. Increasing camber (ψ) is therefore preferable to increasing angle of attack (α) in order to avoid leading edge separation and stall.

7.

(a) Infinite reflections are needed.



Each image system has span $2s'$ where $s' = \frac{\pi}{4} s$.

It is less accurate to use $s' = s$, as explained in notes.

(b) The image system does not affect the streamwise velocity at the control point P. Therefore the relative velocity remains U and α does not change. Hence α does not change. [Does change in apparent angle of attack change lift?]

(c) For each semi-infinite vortex, $w = \frac{\Gamma}{4\pi d}$. Therefore calculate the downwash at P.

Before entering the canyon, $w_c = 2\left(\frac{-\Gamma}{4\pi s'}\right) = -\frac{\Gamma}{2\pi s'}$

Inside the canyon,
$$w_c = 2\left\{-\frac{\Gamma}{4\pi}\left(\frac{1}{s'} \underbrace{- \frac{1}{2b-s'} + \frac{1}{2b+s'}} - \frac{1}{4b-s'} + \frac{1}{4b+s'} - \dots\right)\right\}$$
$$= 2\left\{-\frac{\Gamma}{4\pi}\left(\frac{1}{s'} + \frac{(2b-s')-(2b+s')}{4b^2-s'^2} + \frac{(4b-s')-(4b+s')}{16b^2-s'^2} + \dots\right)\right\}$$
$$\Rightarrow \Delta w_c = \frac{\Gamma s'}{\pi}\underbrace{\left(\frac{1}{4b^2-s'^2} + \frac{1}{16b^2-s'^2} + \frac{1}{36b^2-s'^2} + \dots \frac{1}{(2nb)^2-s'^2} \dots\right)}$$

Drag : $D_i = 2\rho\Gamma s' w \Rightarrow \frac{\Delta D_i}{D_{i,0}} = \frac{\Delta w}{w_c} = \frac{\Gamma s'/\pi}{\Gamma/2\pi s'}\left(\dots\right) = 2s'^2\left(\dots\right) = \sum_{n=1}^{\infty} \frac{2}{\left(2n\frac{b}{s'}\right)^2 - 1}$

The relative change in induced drag coefficient is $\sum_{n=1}^{\infty} \frac{2}{(2n\frac{b}{s'})^2 - 1}$ where $\frac{b}{s'} = \frac{4b}{\pi s} = \frac{8}{\pi}$ when $b = 2s$.

$$\Rightarrow \frac{\Delta D_i}{D_{i,0}} = \sum_{n=1}^{\infty} \frac{2}{(\frac{16n}{\pi})^2 - 1} = 2\left(0.04010 + 9.73\times10^{-3} + 4.30\times10^{-3} + 2.41\times10^{-3} + \dots\right)$$
$$\simeq 0.11$$

Therefore there is an approximate 11% reduction in induced drag

d) If $b = s$, $\frac{\Delta D_i}{D_{i,0}} = \sum_{n=1}^{\infty} \frac{2}{(\frac{8n}{\pi})^2 - 1} = 2\left(0.1823 + 0.0401 + 0.0174 + 9.73\times10^{-3}\right)$
$$\simeq 0.5$$

This gives a ≈ 50% reduction in induced drag. But this ignores the fact that, for s=b, the wing effectively becomes 2D, for which the induced drag is zero. This disagreement shows that the horseshoe vortex model is quite crude, especially in the way it models roll-up of the tip vortex.

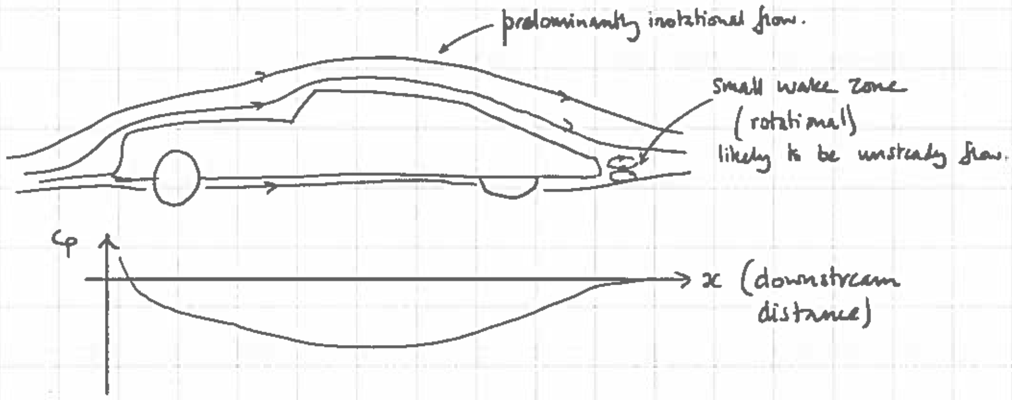
8 (a) The rear of the car is tapered over a long distance, which will cause the wake to be small.

The front is also streamlined, without any large projections into the flow.

The rear wheels are well-shrouded.

There are no wing mirrors.

(b)



(c) Cavities can be found:

- between cab and trailer on a lorry
- on the undersides of cars
- on the tops of trains
- when a car window or sun roof opens

The flow separates at the upstream corner of the cavity. There is a shear layer between the cavity flow and the external flow. Kelvin–Helmholtz vortices form. These travel downstream and impinge on the downstream corner of the cavity. This generates an acoustic wave, which travels upstream and triggers the next K-H vortex. This causes noise. At low Mach number, this can also trigger the Helmholtz mode within the cavity (e.g. the car) causing an uncomfortable noise for the passengers.

(d) Small scale features:

- reduce ridges on the car through improved manufacturing techniques. This reduces drag and noise.
- reduce gaps and cavities on the top and underside of the car.
  - e.g. use a diffuser profile shape between tank and bumper.
- carefully round the A and C pillars.
- shield the wheels from the flow
- deflect the flow away from the windscreen wipers.
- choose the spoiler gap carefully.

(e) Assume $C_D = 0.27$ for the car and $C_D = 1.0$ for the mirror.

$F_{car} = 0.27\left(\frac{1}{2}\rho U^2 A_{car}\right)$

$F_{mir} = 1.0\left(\frac{1}{2}\rho U^2 A_{mir}\right)$ assuming that the wake of the mirror has the same frontal area as the mirror itself.

$\Rightarrow \frac{F_{mir}}{F_{car}} = \frac{1.0}{0.27}\frac{A_{mir}}{A_{car}} = 0.037 = 3.7\%$

This is a significant contributor to overall drag. Designers must keep appendages such as these to a minimum.

(f) The underside of a car is typically populated by bluff components and is a zone with poor aerodynamics. This creates substantial drag and noise. Through careful nose design, the stagnation point can be moved downwards. This deflects more air upwards, meaning that less flow goes under the car, hence improving drag and noise. Placing the radiator in the expected stagnation zone will force more air through it and therefore improve cooling.