3A3 - 2015

Q1) a) h_0 is constant along streamline when there is no <u>work</u> or <u>heat transfer</u>, because:

$$\dot{Q} - \dot{W}_x = \dot{m}(\Delta h + \Delta e_{kin}) = \dot{m}(\Delta h_0)$$

(we are also neglecting changes in potential energy)

b) $c_p T_0 = h_0 = h + \frac{1}{2}V^2 = c_p T + \frac{1}{2}V^2$

Perfect gas: $c_p = \text{const.}$

$$V = aM = \sqrt{\gamma RT}\, M$$

$$\therefore \quad c_p T_0 = c_p T + \frac{1}{2}\gamma RT M^2$$

$$\therefore \quad \frac{T_0}{T} = 1 + \frac{\gamma R}{2 c_p} M^2 \qquad \text{use: } R = c_p - c_v \text{ and } \gamma = \frac{c_p}{c_v}$$

$$\therefore \quad \underline{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2}$$

for isentropic flow $\frac{T_0}{T} = \left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}$ (databook)

hence: $\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$

for $M \rightarrow 0$, use series expansion:

$$\frac{p_0}{p} = 1 + \frac{\gamma}{\gamma-1} \cdot \frac{\gamma-1}{2} M^2 + \cancel{C \cdot M^4 + \ldots} \rightarrow \approx 0$$

$$\frac{p_0}{p} = 1 + \frac{\gamma}{2} M^2 + O(M^4)$$

$$p_0 = p + \frac{1}{2}\gamma p M^2 + O(M^4) = p + \frac{1}{2}\gamma \cdot p \cdot \frac{V^2}{\gamma RT} + O(M^4)$$

Thus for $M \rightarrow 0$ $\quad p_0 = p + \frac{1}{2}\rho V^2$

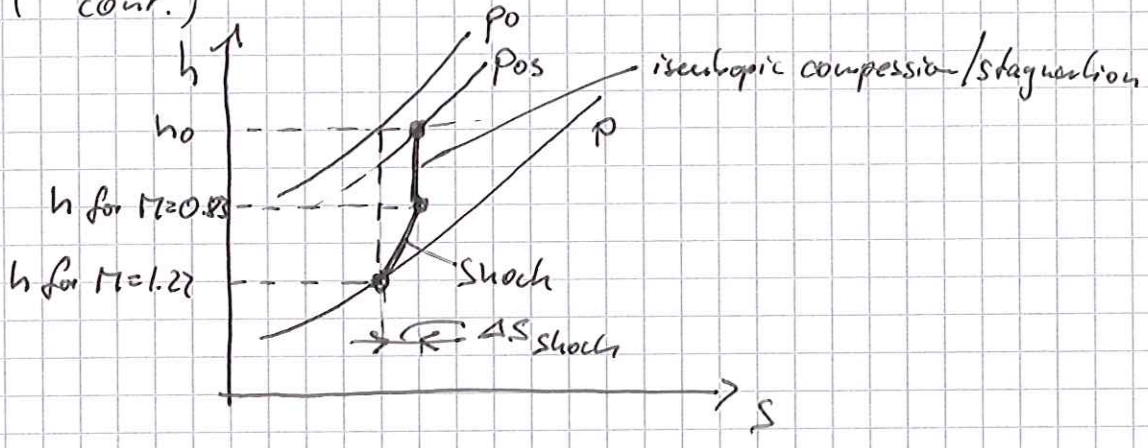
c) At 9000 m: (databook) $\quad T = 229.74\,K \quad p = 30803\,Pa \quad a = 303.6\,\frac{m}{s}$

$V = 370\,\frac{m}{s} \Rightarrow$ shock ahead of Pitot. $(M = 1.22)$

Thus: $\frac{p}{p_0} = 0.4017$ (databook $M = 1.22$) $\qquad \frac{p_{0s}}{p_0} = 0.9907$ (databook)

Thus: $\underline{p_{0s} = 75\,970\,Pa}$

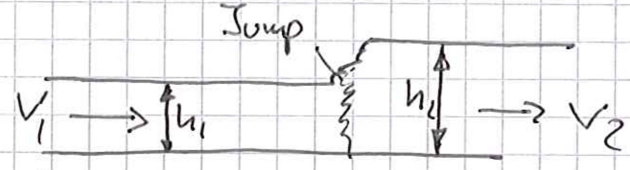
Q1 cont.)



h
p0
p0s
isentropic compession/stagnation
h0
p
h for M=0.83
h for M=1.27
Shock
ΔS shock
S

Q2

a)



per unit width:

Continuity: $V_1 h_1 = V_2 h_2$ $\quad (\dot m = \rho V_1 h_1 = \rho V_2 h_2)$

Momentum: $(\Sigma pA)_x + 0 = \dot m (V_2 - V_1)$

$$\frac{1}{2}\rho g h_1^2 - \frac{1}{2}\rho g h_2^2 = \rho V_1 h_1 V_2 - \rho V_2 h_2 V_1$$

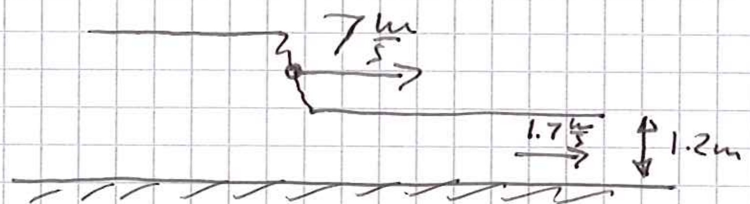
$$\Rightarrow \frac{1}{2} g (h_1^2 - h_2^2) = V_1 V_2 (h_1 - h_2)$$

Trivial solution $h_1 = h_2$

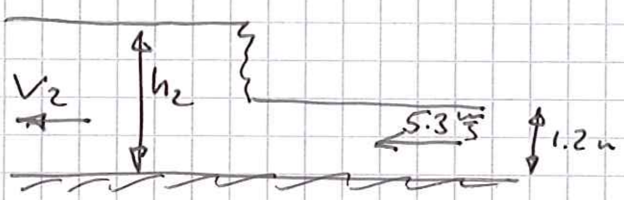
Jump solution $\frac{1}{2} g (h_1 + h_2) = V_1 V_2 = V_1^2 \cdot \frac{h_1}{h_2}$

$$Fr^2 = \frac{V_1^2}{g h_1} = \frac{1}{2}\frac{h_2}{h_1^2}(h_1 + h_2) = \underline{\underline{\frac{1}{2}\frac{h_2}{h_1}\left(1 + \frac{h_2}{h_1}\right)}}$$

b) Stationary frame of ref:



In jump frame of ref:



$Fr_1 = \frac{V_1^2}{g h_1} = 2.386$

set $X = \frac{h_2}{h_1}$ $\quad \therefore \quad X^2 + X - 2 \cdot Fr_1^2 = 0$

$$X_{1/2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4 \cdot Fr_1^2} = 1.741 \text{ (+ solution)}$$

Thus: $h_2 = 2.09\,m$ $\qquad V_2 = V_1 \frac{h_1}{h_2} = 3.044 \frac{m}{s}$

in stationary frame: $V_2 = 7 - 3.044 = 3.956 \frac{m}{s}$

Q2 c) Wave speeds: $\sqrt{gh}$

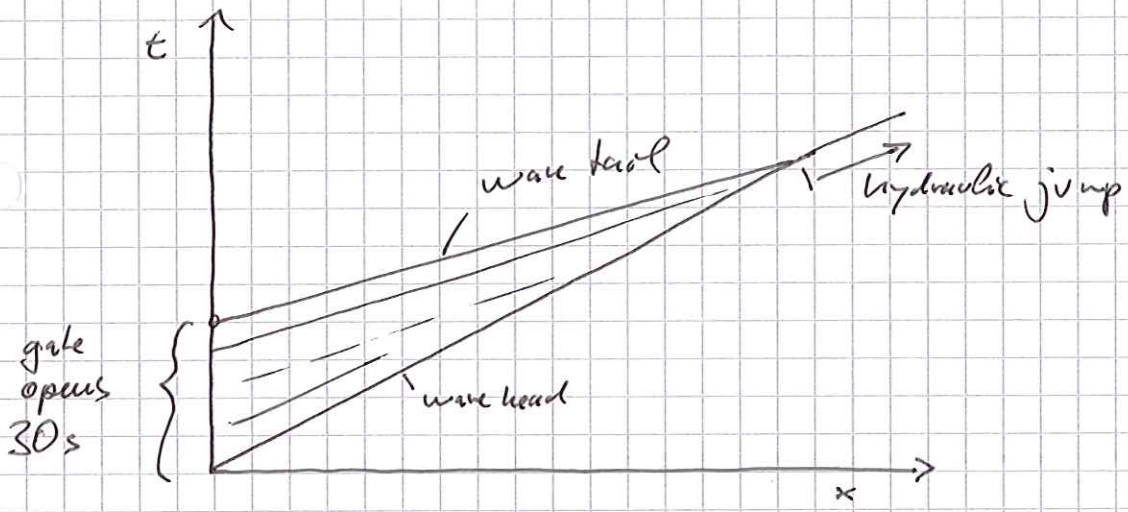
$\Rightarrow$ head wave $\sqrt{gh_1} = 3.83 \frac{m}{s} = c_1$

tail wave $\sqrt{gh_2} = 4.528 \frac{m}{s} = c_2$

add flow velocity

$V_1 + c_1 = 5.13 \frac{m}{s}$

$V_2 + c_2 = 8.48 \frac{m}{s}$



Time to coalesce all waves:

$5.13 \frac{m}{s} \cdot t = 8.48 \frac{m}{s} (t - 30s)$

$\Rightarrow \underline{t = 75.9 \text{ s}}$

Distance $d = 5.13 \frac{m}{s} \cdot 75.9s = \underline{\underline{389.4 \text{ m}}}$

3) a) Impulse function

$$F = A(p + \rho V^2)$$

Thus $\frac{F}{A} = p + \rho V^2 = p + \rho V \cdot V$

Continuity: $\rho V = \text{const.}$ (hence $\delta(\rho V) = 0$ $\therefore$ $\rho \delta V + V \delta \rho = 0$)

Thus: $\delta\left(\frac{F}{A}\right) = \delta p + \rho V \delta V + \underbrace{\cancel{V \delta(\rho V)}}_{=0}$

using energy eqn: $h + \frac{1}{2}V^2 = \text{const.}$ (hence: $dh + V dV = 0$)

ideal gas eqn: $p = \rho R T$ and $h = c_p T$

Thus: $p = \rho \frac{R}{c_p} h = \frac{\gamma - 1}{\gamma} \rho h$

hence: $\delta\left(\frac{F}{A}\right) = \frac{\gamma-1}{\gamma} \cdot \rho \delta h + \frac{\gamma - 1}{\gamma} h \delta \rho + \rho V \delta V$

$= -\frac{\gamma-1}{\gamma} \cdot \rho V \delta V + \frac{p}{\rho} \delta \rho + \rho V \delta V$

$= \frac{1}{\gamma} \cdot \rho V \delta V + p \frac{\delta \rho}{\rho} = \frac{\rho V}{\gamma} \delta V - p \frac{\delta V}{V}$

$= p \cdot M^2 \frac{\delta V}{V} - p \frac{\delta V}{V} = \underline{\underline{p(M^2 - 1)\frac{\delta V}{V}}}$

b) Due to friction $\delta\left(\frac{F}{A}\right) < 0$

Thus: $M < 1$: $\frac{\delta V}{V} > 0$ Acceleration

$M > 1$: $\frac{\delta V}{V} < 0$ Deceleration

c) i) $\frac{4 c_f}{D} = \frac{4 \cdot 0.0025}{0.2} = 0.05$ $L = L_{max} \Rightarrow \frac{4 c_f L}{D} = 0.136$

Tables: $M = 1.5$

ii) from tables: $\frac{\dot{m}\sqrt{c_p T_0}}{A p_0} = 1.0891$

hence $\dot{m} = \frac{1.0891 \cdot \pi \frac{D^2}{4} p_0}{\sqrt{c_p T_0}} = 12.5 \frac{kg}{s}$

3 cont.) Length to shock wave $L = L_{max} - 0.438m$

$= 2.282m$

Thus $\frac{4C_fL}{D} = 0.1141$

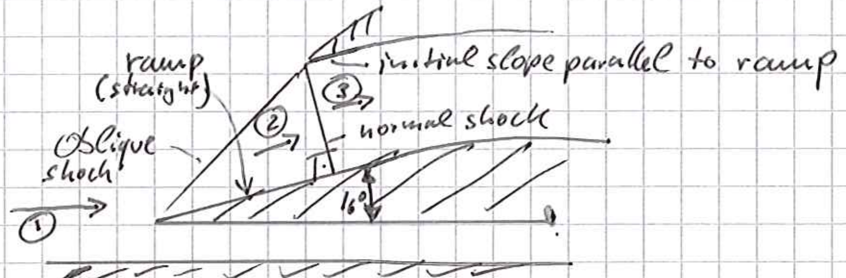
from Tables: $M = 1.44$ thus $M_s = 0.7235$

behind shock ($M_s = 0.7235$) $\frac{4C_fL}{D} = 0.1665$

hence $L = 3.33$ m

thus $L_{total} = (3.33 + 0.438)m = \underline{\underline{3.768m}}$

4) a)



The gap acts as boundary layer bleed which helps to:

- reduce the amount of low-momentum fluid in inlet
- improves the resistance of the ramp boundary layer to shock-induced separation

Both improve inlet performance / reduce losses, but there is additional drag from splitter.

b) Weak oblique shock. $M = 2$, $\Theta = 16^\circ \Rightarrow M_2 = 1.4034$ $\frac{p_{02}}{p_{01}} = 0.943$ (Tables)

Normal shock $M = 1.4$ $\frac{p_{03}}{p_{02}} = 0.9582$

(or, interpolate for $M = 1.4034$, to give 0.9574)

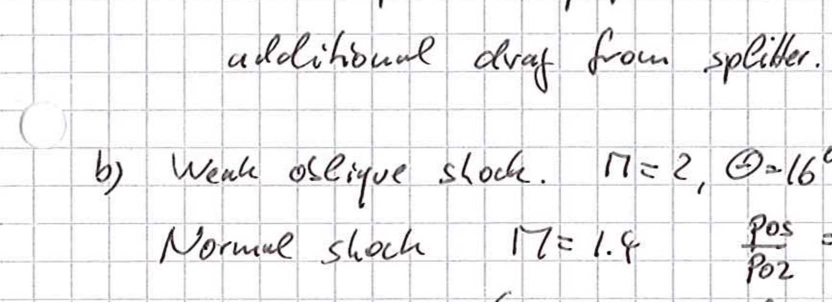
Thus: $\frac{p_{03}}{p_{01}} = \underline{0.903}$

c) At $M = 1.8$: $\Theta = 16^\circ$ $\beta = 53.198^\circ$ $M_2 = 1.1958$ $\frac{p_{02}}{p_{01}} = 0.9473$

Normal shock : $M = 1.1958$ $\frac{p_{03}}{p_{02}} = 0.9936$

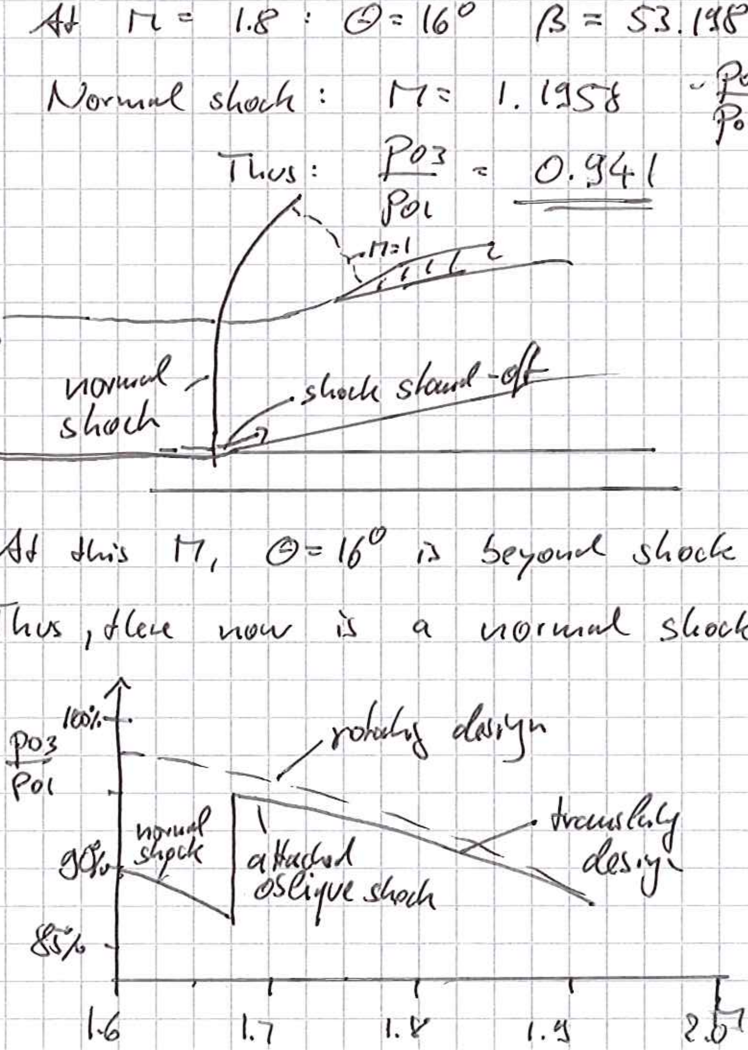
Thus: $\frac{p_{03}}{p_{01}} = \underline{\underline{0.941}}$

d)



At this M, $\Theta = 16^\circ$ is beyond shock detachment criterion.
Thus, there now is a normal shock wave ($p_{03}/p_{01} = 0.895$)

e)



5)

a) From data book:

$$\tan\Theta = \frac{2\cot\beta(M_1^2\sin^2\beta - 1)}{(\gamma+1)M_1^2 - 2(M_1^2\sin^2\beta - 1)}$$

for $M_1 \to \infty$: $\tan\Theta \to \frac{2\cot\beta \; M_1^2\sin^2\beta}{(\gamma+1)M_1^2 - 2M_1^2\sin^2\beta} = \frac{2\cos\beta\sin\beta}{(\gamma+1) - 2\sin^2\beta}$

thus $\underline{\Theta \to \arctan \frac{\sin 2\beta}{\gamma+1 - 2\sin^2\beta}}$

b) Note: $\frac{\sin 2\beta}{\gamma+1 - 2\sin^2\beta} = \frac{\sin 2\beta}{\gamma + \cos 2\beta}$

Thus: $\frac{d(\tan\Theta)}{d\beta} = \frac{2\cos 2\beta}{\gamma + \cos 2\beta} - \frac{\sin 2\beta(-2\sin 2\beta)}{(\gamma + \cos 2\beta)^2} = 0$

$$2\cos 2\beta(\gamma + \cos 2\beta) + 2\sin^2 2\beta = 0$$

$$2\gamma\cos 2\beta + \underbrace{2\cos^2 2\beta + 2\sin^2 2\beta}_{1} = 0$$

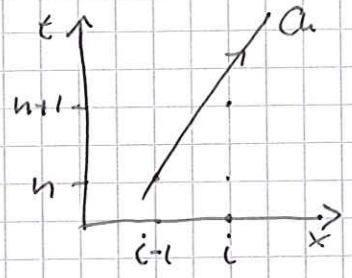
$$\Rightarrow \cos 2\beta = -\frac{1}{\gamma}$$

for $\gamma = 1.4$, maximum for $\beta = 67.8°$, $\underline{\underline{\Theta_{max} = 45.6°}}$

c) The best performance arises when the stag. pressure loss (entropy production) is equally split for all three shocks. One way to look at this is to consider the shock normal Mach number in each case. Since the flow M reduces across each shock wave, the deflection angle should increase. A single shock has the greatest loss while (b) offers the best solution

The downside is a more complex / expensive and (thus heavier) longer system.

6) a)



Charachristic $\frac{dx}{dt} = A$

i) "Upwinding" takes into account the direction in which information is flowing. Provided the next point is below the characteristic the scheme is stable.

See also lecture notes.

ii) c is the CFL (Courant Friedrichs Levi) or Courant number.

It describes how close the numerical scheme is to the charachistic and thus it describes the stability.

c must be < 1 for stability.

$C = A \cdot \frac{\delta t}{\delta x}$

b) $u_i^{n+1} = u_i^n - c(u_i^n - u_{i-1}^n)$

Taylor exp.: (exact) $u_i^{n+1} = u_i^n + \delta t \frac{\partial u}{\partial t}\Big|_{i,n} + \frac{\delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_{i,n} + \dots$

$$u_{i-1}^n = u_i^n - \delta x \frac{\partial u}{\partial x}\Big|_{i,n} + \frac{\delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_{i,n} - \dots$$

Thus $\frac{\partial u}{\partial t} = \frac{1}{\delta t}(u_i^{n+1} - u_i^n) - \frac{\delta t}{2}\frac{\partial^2 u}{\partial t^2} + O(\delta t^3)$

and $\frac{\partial u}{\partial x} = \frac{1}{\delta x}(u_i^n - u_{i-1}^n) + \frac{\delta x}{2}\frac{\partial^2 u}{\partial x^2} + O(\delta x^3)$

into convection eqn:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \frac{1}{\delta t}(u_i^{n+1} - u_i^n) - \frac{\delta t}{2}\frac{\partial^2 u}{\partial t^2} + A\frac{1}{\delta x}(u_i^n - u_{i-1}^n) + A\frac{\delta x}{2}\frac{\partial^2 u}{\partial x^2} + O(^3)$$

$$= A\frac{\delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\delta t}\Big[\underbrace{u_i^{n+1} - u_i^n + A\frac{\delta t}{\delta x}(u_i^n - u_{i-1}^n)}_{=0}\Big] + O(^3)$$

6 b) cont.

using: $\frac{\partial u}{\partial t} = -A\frac{\partial u}{\partial x}$ hence $\frac{\partial^2 u}{\partial t^2} = A^2\frac{\partial^2 u}{\partial x^2}$

gives $$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = A\frac{\delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\delta t}{2}A^2\frac{\partial^2 u}{\partial x^2} + O(\ ^3)$$

$$= A\frac{\delta x}{2}\left(1 - A\frac{\delta t}{\delta x}\right)\frac{\partial^2 u}{\partial x^2} + O(\ ^3)$$

Since $\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x}$ is $= 0$ numerical errors are:

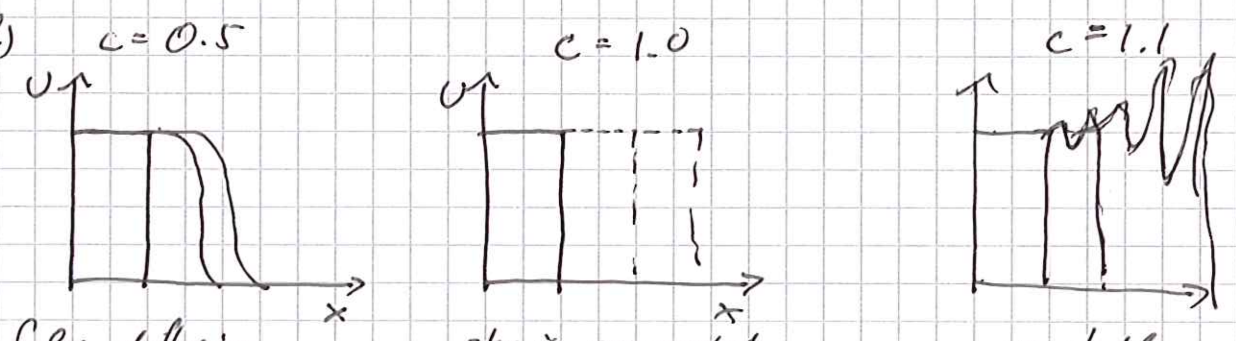
$$A\frac{\delta x}{2}\left(1 - A\frac{\delta t}{\delta x}\right)\frac{\partial^2 u}{\partial x^2} + O(\ ^3)$$

which is a diffusion term with a coefficient

$$A\frac{\delta x}{2}\left(1 - A\frac{\delta t}{\delta x}\right) = \nu_{eff}$$ "numerical viscosity"

c) from above, error is first order in δx and δt (space + time).

d)



false diffusion

step is propagated accurately ($\nu_{eff} = 0$ at $c = 1$)

unstable

Q7, a)

i) $\frac{\partial \rho}{\partial t} + \nabla \rho \underline{u} = 0$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \rho \underline{u} \right) dV = 0$$

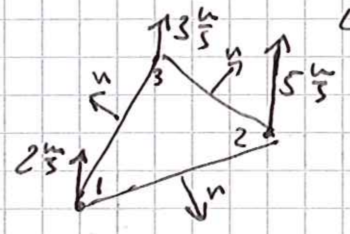
Apply Gauß: $\int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho \underline{u} \underline{n} \, dA = 0$

$\frac{\partial \bar{\rho}}{\partial t} \Delta V + \int_A \rho \underline{u} \underline{n} \, dA = 0$ where $\bar{\rho}$ is volume average of ρ
$\bar{n}$ is normal vector

In 2-d: $\frac{\partial \bar{\rho}}{\partial t} \cdot A + \int_L \rho \underline{u} \underline{n} \, dL$ L: circumference

for a cell: A is cell area
$\int_L \rho \underline{u} \underline{n} \, dL$ is mass flow out of cell

b)



$\rho = 1.0 \frac{kg}{m^3}$

$$\int_L \rho \underline{u} \underline{n} \, dL = \underbrace{\int \rho u n_x dL}_{=0} + \int \rho v n_y dL$$

Segment $1 \to 2$: $n_y dL = -5 \cdot 10^{-4} m$ (from Geometry: projected area)

$2 \to 3$: $n_y dL = 3 \cdot 10^{-4} m$

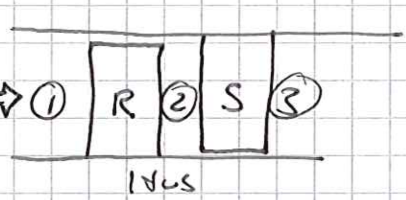
$3 \to 1$: $n_y dL = 2 \cdot 10^{-4} m$

Thus: $\int \rho v n_y dL = \rho \cdot \left[-\frac{V_1 + V_2}{2} \cdot 5 + \frac{V_2 + V_3}{2} \cdot 3 + \frac{V_3 + V_1}{2} \cdot 2 \right] \cdot 10^{-4} m$

$= -5 \cdot 10^{-5} \frac{kg}{s}$ Thus: $\underline{\dot{m}_{in} = 5 \cdot 10^{-5} \frac{kg}{s}}$

Q7 b) Incompressible + const. area

$\Rightarrow V_x = \text{const.}$



i) $\Delta h_0 = U(V_{\theta 2} - V_{\theta 1})$

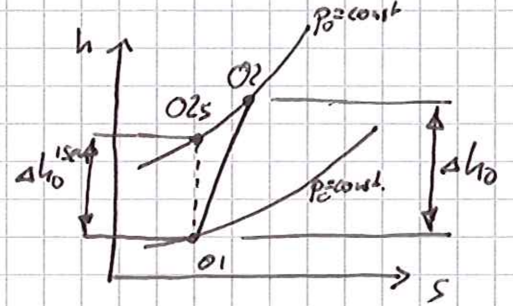
where $V_\theta = V_{\theta_{rel}} + U$

Thus: $\Delta h_0 = U\left(V_{\theta 2 rel} + U - V_{\theta_1}\right)$

$$\frac{\Delta h_0}{U^2} = \frac{V_{x2}}{U}\tan\alpha_2^{rel} + 1 - \frac{V_{x1}}{U}\tan\alpha_1$$

here $\alpha_1 = 0$; $\phi = \frac{V_{x2}}{U}$ $\Rightarrow$ $\frac{\Delta h_0}{U^2} = 1 + \phi\tan\alpha_{2,rel}$

ii) $\eta_{tt} = \frac{\Delta h_0^{isen}}{\Delta h_0}$

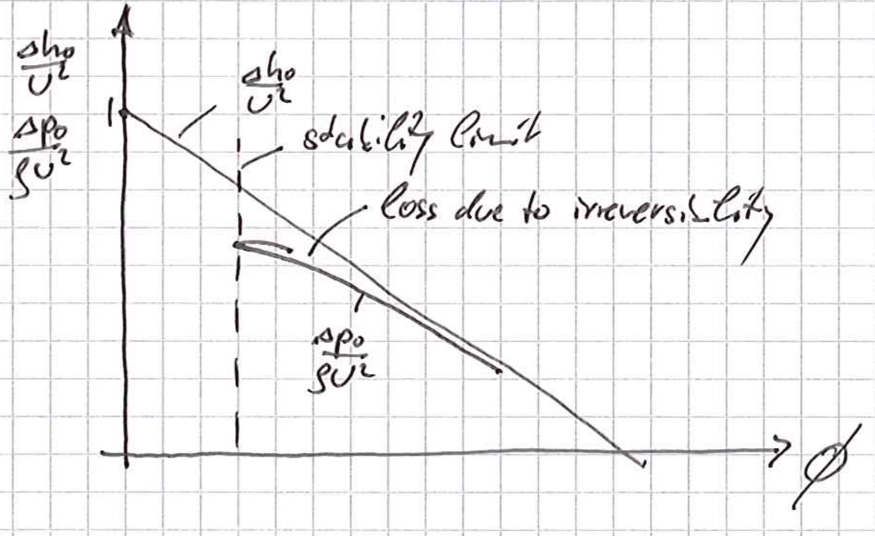


$T\Delta s = \Delta h_0 - \frac{1}{\rho}\Delta p_0$ (incompressible)

and $\Delta h_0^{isen} = \frac{1}{\rho}\Delta p_0$

thus: $\eta_{tt} = \frac{\frac{1}{\rho}\Delta p_0}{\Delta h_0} = \frac{\frac{1}{\rho}\frac{\Delta p_0}{U^2}}{\frac{\Delta h_0}{U^2}}$ $\therefore$ $\frac{\Delta p_0}{\rho U^2} = \eta_{tt}\frac{\Delta h_0}{U^2}$

iii)



8) a) $\Delta h_0 = U(V_{\theta 2} - V_{\theta 1}) = c_p(T_{02} - T_{01})$

$V_{\theta 1} = 0 \quad \Rightarrow \quad V_{\theta 2} = \dfrac{c_p(T_{02} - T_{01})}{U} = \underline{106.2 \ \tfrac{m}{s}}$

b) Relative turning: $V_\theta^{rel} = V_\theta - U$

$\tan\alpha_1 = \dfrac{V_{\theta 1}^{rel}}{V_{x1}} = \dfrac{0 - U}{V_{x1}} = -\dfrac{265}{132} \Rightarrow \alpha_1^{rel} = -63.5^0$

$\tan\alpha_2 = \dfrac{V_{\theta 2}^{rel}}{V_{x2}} = \dfrac{106.2 - 265}{118} \qquad \alpha_2^{rel} = -53.4^0$

$\therefore$ Flow turning: $\underline{\underline{10.1^0}}$ $\underline{\text{towards}}$ the axial direction

c) $\eta_{tt} = \dfrac{\Delta h_0^{ISEN}}{\Delta h_0} = \dfrac{\Delta T_0^{ISEN}}{\Delta T_0} \Rightarrow \Delta T_0^{ISEN} = \eta_{tt}\,\Delta T_0$

$\Delta T_0^{ISEN} = 0.95(308 - 280) = 26.6K \Rightarrow T_{02}^{ISEN} = 280 + 26.6 = \underline{\underline{306.6K}}$

$\dfrac{p_{02}}{p_{01}} = \left(\dfrac{T_{02}^{ISEN}}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} = \underline{1.374}$

d) $T_1 = T_{01} - \dfrac{V_{x1}^2}{2c_p} = 271.3K$

$\dfrac{p_1}{p_{01}} = \left(\dfrac{T_1}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} = 0.8954$

$T_2 = T_{02} - \dfrac{(V_{x2}^2 + V_{\theta 2}^2)}{2c_p} = 295.5K$

$\dfrac{p_2}{p_{02}} = \left(\dfrac{T_2}{T_{02}}\right)^{\frac{\gamma}{\gamma-1}} = 0.865$

$\dfrac{p_2}{p_1} = \dfrac{p_2}{p_{02}} \cdot \dfrac{p_{02}}{p_{01}} \cdot \dfrac{p_{01}}{p_1} = \underline{1.327}$

e) $\dfrac{\rho_2}{\rho_1} = \dfrac{p_2}{RT_2} \cdot \dfrac{RT_1}{p_1} = \dfrac{p_2}{p_1}\dfrac{T_1}{T_2} = 1.218$

Hence $\dfrac{A_2}{A_1} = \dfrac{\rho_1 V_{x1}}{\rho_2 V_{x2}} = \underline{0.918}$

Area decrease to offset density rise.

But, velocity has dropped so area should be decreased $\underline{\text{further}}$ (to avoid low V_x which gives high α).

8 f)

$$Y_p^{stator} = \frac{p_{02} - p_{03}}{p_{02} - p_2} = 0.08 \qquad \text{Note } p_{03s} = p_{02}$$

Thus:

$$\frac{p_{03}}{p_{02}} = 1 - 0.08\left(1 - \frac{p_2}{p_{02}}\right)$$

$$\frac{p_{03}}{p_{02}} = 0.9892$$

$$\frac{p_{03}}{p_{01}} = \frac{p_{03}}{p_{02}} \cdot \frac{p_{02}}{p_{01}} = 0.9892 \cdot 1.374 = \underline{\underline{1.359}}$$

3A3 Fluid Mechanics II - 2015

Assessor's comments

Q1 Compressible stagnation pressure
This was a very popular question which was generally very well done. Some candidates did not realise that there is a shock ahead of a Pitot tube in supersonic flow.

Q2) Hydraulic jump
A popular question with many good answers. A number of candidates did not explicitly state that no force is present when applying the SFME (or say why) and a further common mistake was to forget to add the flow speed to the wave speeds in part c)

Q3) Adiabatic flow with friction
The most popular question of the examination, chosen by almost all candidates. This question also achieved the highest average score. Some candidates did not realise that the flow is supersonic in part c) and several did not ciorrectly account for the shock in part d).

Q4) Supersonic inlet
Another popular question with many good answers. Many candidates could not draw the correct flow diagram when the inflow M was reduced (stand-off normal shock wave) in part c). In part d) only very few candidates correctly understood the mode-switch at M=1.656 when the leading shock wave changes to an oblique shock.

Q5) Oblique shock wave behaviour
A popular question that was generally well answered. However, many candidates found the calculation of the maximum turning angle in part b) difficult. Only few could reason why a weak shock followed by a stronger one would give the best performance – nonlinearity or entropy generation was rarely considered.

Q6) Scalar convection equation
A very unpopular question that was however very well answered by those that attempted it.

Q7) Numerical solution of continuity equation and single stage axial flow compressor
Very unpopular question, attempted by only a handful of candidates. Many candidates struggled to draw a correct diagram in part b) iii) and none included the stability limit.

Q8) Axial compressor stage
A reasonably popular question attempted by about half of all candidates. Although there were some good answers many candidates clearly ran out of time and could not complete the question, which resulted in a low average mark. A very common mistake was to give the absolute flow angle in part b) rather than the flow turning across the stage. Not many candidates could comment on the physical significance of the result in e)

HB 2015

3A3 Fluid Mechanics II – 2015

Numerical answers

Q1

c) 75.97kPa

Q2

b) h2=2.09m, v2=3.956m/s

c) t=75.9s, d=389.4m

Q3

c) i) M=1.5 ii) mass flow rate = 125kg/s

d) M=1.44, L=3.768m

Q4

b) p0s/p01=0.903

c) p0s/p01=0.941

d) p0s/p01≈0.895

Q5

b) Theta_max=45.6°

c) i) worst, iii) best

Q6

c) First order accurate in time and space (unless c=1)

Q7

a) ii) m_dot=5x10-5kg/s

Q8

a) 106.2m/s

b) 10.1°

c) p02/p01=1.374

d) p2/p1=1.327

e) A2/A1=0.918

f) p03/p01=1.359

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