Q1

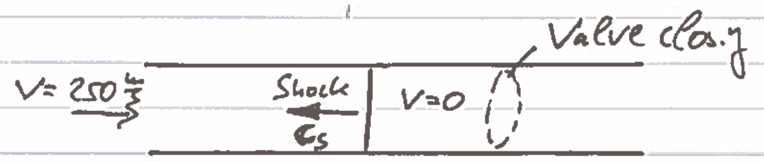
a) from data book

$$M_2 = \sqrt{\frac{1 + \frac{\gamma-1}{2}M^2}{\gamma M^2 - \frac{\gamma-1}{2}}}$$

M_2: M downstream of shock wave

for $M \to \infty$ $\quad M_2^2 \Rightarrow \dfrac{\frac{\gamma-1}{2}M^2}{\gamma M^2} = \dfrac{\gamma-1}{2\gamma}$ (actually this is a minimum)

b)



In shock reference frame (shock moves at c_s):

$$\xrightarrow{c_s + V} \; \big| \; \xrightarrow{c_s}$$
$$p, \rho \; \big| \; p_s, \rho_s$$

Continuity: $\rho_s c_s = \rho(c_s + V)$ $\qquad A = \text{const.}$

thus $\dfrac{\rho_s}{\rho} = \dfrac{c_s+V}{c_s} = \dfrac{(\gamma+1)M_s^2}{2(1+\frac{\gamma-1}{2}M_s^2)}$ (Databook) where M_s = shock Mach number)

also: $M_s = \dfrac{c_s + V}{a}$

using $M = \dfrac{V}{a}$ (inflow Mach number in stationary frame)

$$\frac{c_s+V}{c_s} = \frac{M_s}{M_s - M} = \frac{(\gamma+1)M_s^2}{2(1+\frac{\gamma-1}{2}M_s^2)}$$

$$\therefore \quad 2 + (\gamma-1)M_s^2 = (\gamma+1)M_s^2 - (\gamma+1)M_s M$$

$$2M_s^2 - (\gamma+1)M_s M - 2 = 0$$

$$\therefore \quad M_s = \frac{\gamma+1}{4}M \pm \sqrt{1 + \left(\frac{\gamma+1}{4}M\right)^2}$$

ignore negative solution (not physical) $\quad T = 500\,\text{K}$; $V = 250\,\frac{m}{s} \Rightarrow a = 448.2\,\text{m/s}$

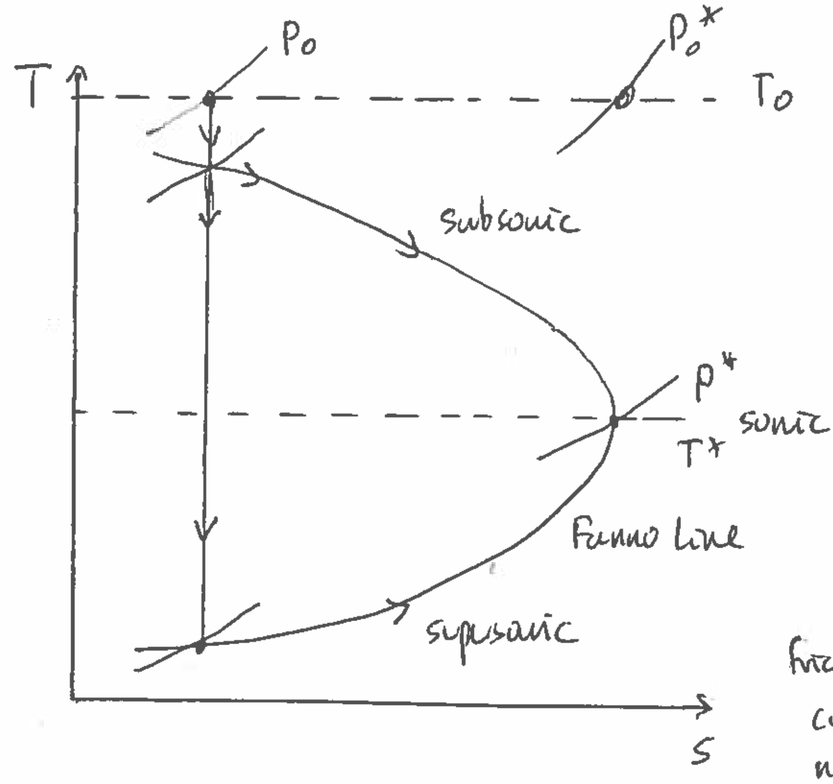
$$M = \frac{250}{448.2} = 0.558$$

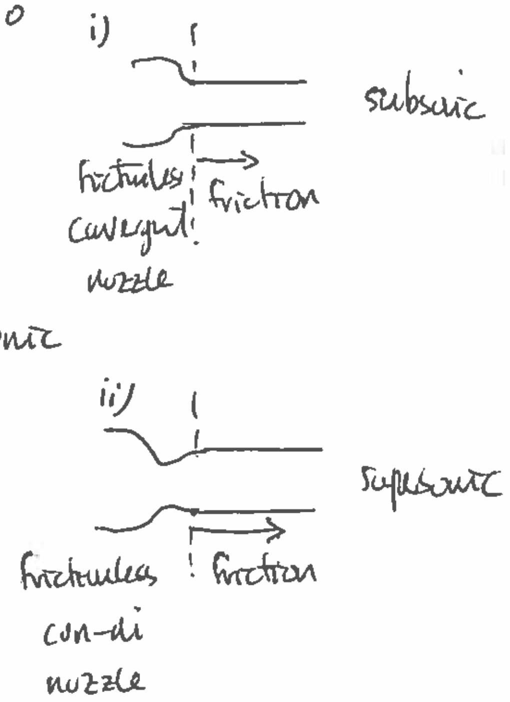
i) $\therefore M_s = 1.389$ $\qquad \therefore c_s = M_s \cdot a - V = 372.5\,\frac{m}{s}$

ii) $\dfrac{p_s}{p} = 2.0875$ (Databook, $M = 1.39$) $\qquad p_s = 313.1\,\text{kPa}$

Note: Iterative solution using tables gives the same result

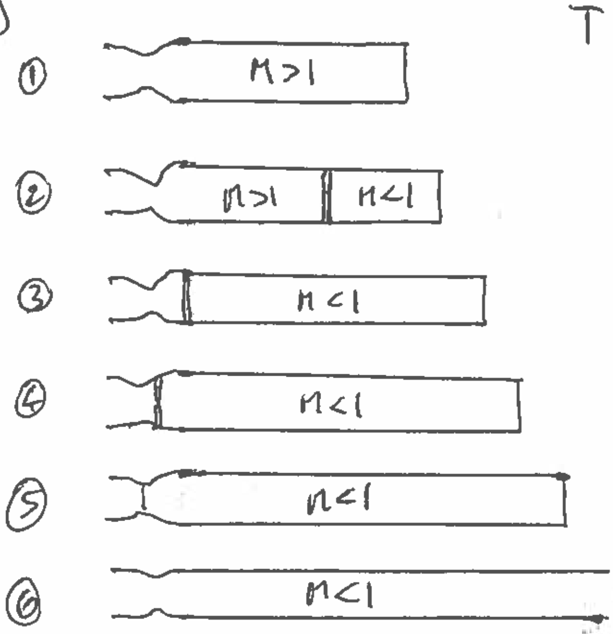
Q2 (a)

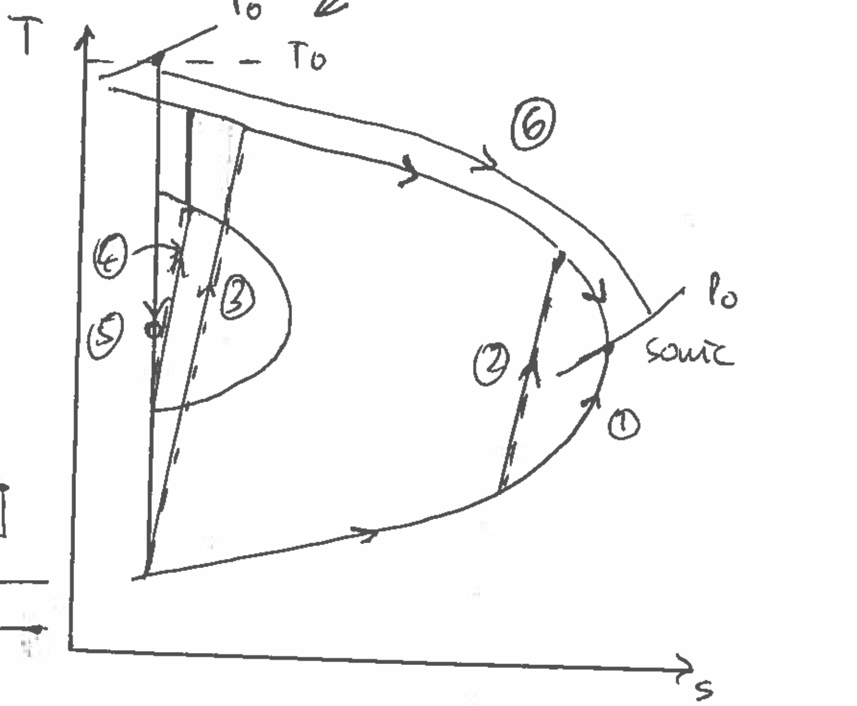




Note curve for shock in nozzle (case 4). Isentropic after shock!

(b)





① supersonic at nozzle exit, pipe exit choked

② " " , shock wave in pipe

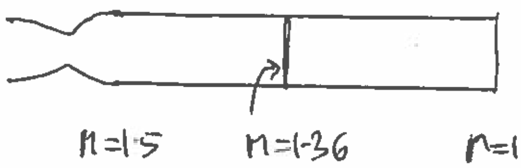
③ shock at nozzle exit, "

④ shock in the nozzle, "

⑤ sonic at nozzle throat, "

⑥ subsonic, unchoked at pipe exit

(c)



$C_f = 0.0025$

Data Book p10 $\quad M = 1.5, \quad \frac{4C_f L_1}{D} = 0.1361$

$M = 1.36, \quad \frac{4C_f L_2}{D} = 0.0855$

$$\frac{4C_f L_{12}}{D} = \frac{4C_f L_1}{D} - \frac{4C_f L_2}{D} = 0.0506$$

Data Book p10, $\quad M = 1.36, \quad M_s = 0.7572$

$$\frac{4C_f L_{2'}}{D} = 0.1181 \quad \text{(interpolated)}$$

Total (dimensionless) length $= \frac{4C_f L_{12}}{D} + \frac{4C_f L_{2'}}{D} = 0.1687$

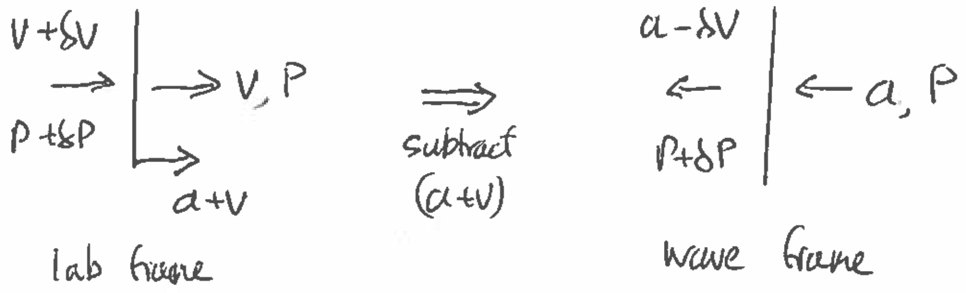
but real length is 5m $\therefore \quad D = \frac{4 \times 0.0025 \times 5}{0.1687}$

ie $\quad \underline{\underline{D = 0.296\,m}}$

Since $\frac{4C_f L_{12}}{D} = 0.0506, \quad L_{12} = 1.498\,m$

<u>Shock is located 1.498 m from the nozzle exit</u>

Q3 (a)



Momentum eqn: $\delta P = \rho a \delta V$

Isentopic relation: $\dfrac{P}{P_0} = \left(\dfrac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} = \left(\dfrac{a}{a_0}\right)^{\frac{2\gamma}{\gamma-1}}$

in differential form: $\dfrac{\delta P}{P} = \dfrac{2\gamma}{\gamma-1}\dfrac{\delta a}{a}$

Speed of sound $a^2 = \dfrac{\gamma P}{\rho}$ giving $2\rho a \delta a = (\gamma-1)\delta P$

Substitute for δP in momentum eqn

$$\delta V - \frac{2}{\gamma-1}\delta a = 0$$

$\therefore\ d\left(V - \dfrac{2}{\gamma-1}a\right) = 0$ and $\underline{V - \dfrac{2a}{\gamma-1} = \text{constant}}$

NB: This part of the question was considered quite hard

(b) (i) Isentropic right-running waves $V - \frac{2a}{\gamma-1} = \text{constant}$

$$\frac{a_2}{a_1} = \left(\frac{T_2}{T_1}\right)^{1/2} = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{\gamma-1}{2}} = (0.6)^{0.2} = 0.903$$

$$V_2 - \frac{2a_2}{\gamma-1} = \overset{=0}{V_1} - \frac{2a_1}{\gamma-1}$$

$$V_2 = \frac{-2a_1}{\gamma-1}\left(1-0.903\right) = -0.485a_1$$

$$a_1 = \sqrt{\gamma R T_1} = 400.9\,m/s \Rightarrow \underline{V_2 = 194.4\,m/s}$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = (0.6)^{1.4} = 0.489 \Rightarrow \underline{P_2 = 97.8\,kPa}$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} = (0.6)^{0.4} = 0.815 \Rightarrow \underline{T_2 = 326.7\,K}$$

(ii) Sonic piston $-V_2 = a_2 \Rightarrow -a_2 - \frac{2a_2}{\gamma-1} = \frac{-2a_1}{\gamma-1}$

$$\frac{a_2}{a_1} = \frac{2}{\gamma+1} = 0.833$$

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{a_2}{a_1}\right)^{\frac{2\gamma}{\gamma-1}} = 0.833^{7} = 0.2783$$

$$\therefore \underline{P_2 = 55.66\,kPa}$$

$$\frac{T_2}{T_1} = \left(\frac{a_2}{a_1}\right)^2 \Rightarrow \underline{T_2 = 277.5\,K}$$

Q4

1a) From databook: $\dfrac{\rho_2}{\rho_1} = \dfrac{(\gamma+1)M_1^2\sin^2\beta}{2\left[1+\frac{\gamma-1}{2}M_1^2\sin^2\beta\right]}$. As $M_1 \to \infty$, $\dfrac{\rho_2}{\rho_1} \to \dfrac{(\gamma+1)M_1^2\sin^2\beta}{(\gamma-1)M_1^2\sin^2\beta} = \dfrac{\gamma+1}{\gamma-1}$

b) From databook: $\tan\theta = \dfrac{2\cot\beta\,(M_1^2\sin^2\beta-1)}{(\gamma+1)M_1^2-2(M_1^2\sin^2\beta-1)}$. As $M_1\to\infty$ $\tan\theta \to \dfrac{2\cot\beta\,(M_1^2\sin^2\beta)}{(\gamma+1)M_1^2-2(M_1^2\sin^2\beta)}$

$$\therefore \tan\theta \to \frac{2\cos\beta\sin\beta}{(\gamma+1)-2\sin^2\beta} = \frac{\sin 2\beta}{\gamma+\cos 2\beta}.$$

c) $$\frac{\partial(\tan\theta_H)}{\partial\beta} = \frac{2\cos 2\beta}{\gamma+\cos 2\beta} - \frac{\sin 2\beta\,(-2\sin 2\beta)}{(\gamma+\cos 2\beta)^2} = 0$$

$$\therefore\quad 2\cos 2\beta\,(\gamma+\cos 2\beta) + 2\sin^2 2\beta = 0$$

$$\gamma\cos 2\beta + \cos^2 2\beta + \sin^2 2\beta = 0 \Rightarrow \cos 2\beta = \frac{-1}{\gamma}$$

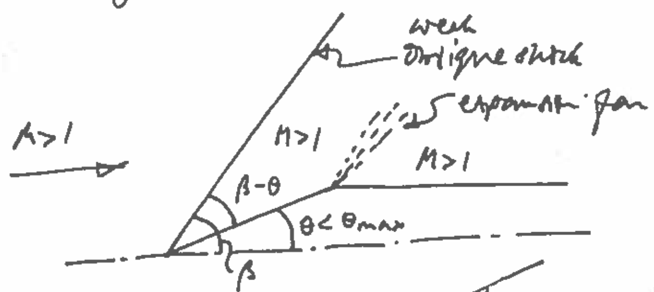
$\therefore$ for $\gamma = 1.4$ $\quad \beta = 67.79°$

$\tan\theta_H = \dfrac{\sin 2\beta}{\gamma+\cos 2\beta}$ $\quad$ for $\beta = 67.79$ $\quad \theta = 45.58°$

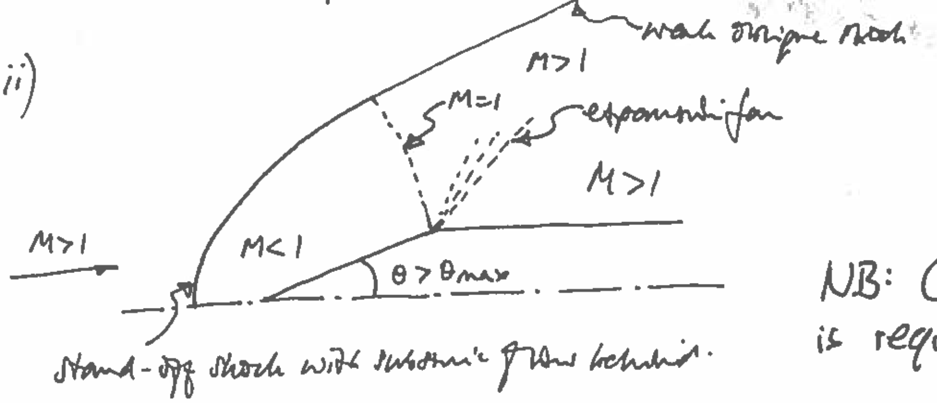
θ_{max} decreases with reducing M

d) Flow will be symmetric around centreline, so consider half:

i)

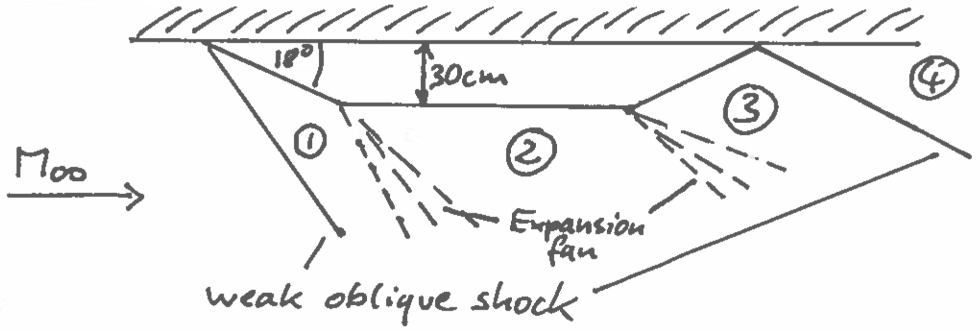


ii)



NB: Careful drawing is required.

Q5



Note: Draw wave angles carefully and note especially their relation to each other. Eg.: Last wave of first expansion fan is parallel to first wave of the second expansion

b) Tables for $\gamma = 1.4$: $M_\infty = 2.00$ $\theta_\infty = 0°$ $\dfrac{p_\infty}{p_{0\infty}} = 0.1278$

$\infty \to ①$: $M_1 = 1.3131$ $\dfrac{p_{01}}{p_{0\infty}} = 0.92092$ $\theta_1 = -18°$ $\nu_1 = 6.527$ $\dfrac{p_1}{p_\infty} = 2.5546$

$① \to ②$: $d(\nu - \theta) = 0$ $\theta_2 = 0°$ $\therefore \nu_2 - \theta_2 = \nu_1 - \theta_1$ $p_{02} = p_{01}$

$\nu_2 = (6.527 + 18) - 0° = 24.527 \Rightarrow M_2 = 1.9335$

$② - ③$: $\theta_3 = +18°$ $\theta_2 = 0°$ $\nu_3 = 42.527 \Rightarrow M_3 = 2.6499$ $\dfrac{p_3}{p_{03}} = 0.04641$

for $M_3 = 2.6499$ $\theta = 18° < \theta_{max}$ $\therefore$ weak oblique ③→④ possible

$\therefore$ do not expect separation.

$\therefore p_1 = 2.5546\, p_\infty$

$p_3 = 0.04641\, p_{03} = 0.04641\, p_{01} = 0.04641 \times 0.92092\, p_{0\infty}$

$= 0.4641 \times 0.92092 \times \dfrac{p_\infty}{0.1278} = 0.3344\, p_\infty$

Drag $= \Delta p \times A = (2.5546 - 0.3344)p_\infty \times 0.3 \times 1 = 0.6661\, p_\infty$ (N)

$h = 18{,}000$ m $\dfrac{p_\infty}{p_{SL}} = 0.0747$ $p_{SL} = 101325$ N/m² $\Rightarrow p_\infty = 7569$ N/m²

$\therefore$ Drag $= 5.042$ kN.

c) Only depth & width affect projected frontal area, A, so increase length.

Note: Length L is not needed for this question!
(and it was wrongly indicated in the figure)

Q6

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \qquad \text{— (A)}$$

(a) Use Taylor series expansion:

$$u_i^{n+1} = u_i^n + \left.\frac{\partial u}{\partial t}\right|_i^n \Delta t + \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n \frac{\Delta t^2}{2} + \cdots \qquad \text{—(1)}$$

$$u_{i-1}^n = u_i^n - \left.\frac{\partial u}{\partial x}\right|_i^n \Delta x + \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n \frac{\Delta x^2}{2} + \cdots \qquad \text{—(2)}$$

The FD scheme is:

$$u_i^{n+1} = u_i^n - c\left(u_i^n - u_{i-1}^n\right) \qquad \text{— (3)}$$

$\Downarrow$

$$\cancel{u_i^n} + \frac{\partial u}{\partial t}\Delta t + O(\Delta t^2) = \cancel{u_i^n} - c\Big(\cancel{u_i^n} - \cancel{u_i^n} + \frac{\partial u}{\partial x}\Delta x - O(\Delta x^2)\Big)$$

$$\therefore \quad \frac{\partial u}{\partial t} + \frac{c\Delta x}{\Delta t}\frac{\partial u}{\partial x} = O(\Delta t^{\cancel{2}}) + O(\Delta x)$$

$$c = \frac{A\Delta t}{\Delta x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = O(\Delta t) + O(\Delta x).$$

The scheme is first order accurate in space and time.

Q6 cont,

(b) Let $u_i^n = \epsilon$ & $u_{i-1}^n = -\epsilon$. Then substituting this in (3) gives:

$$u_i^{n+1} = \epsilon - c(\epsilon + \epsilon)$$

or $$\frac{u_i^{n+1}}{\epsilon} = 1 - 2c$$

for stability,

$$-1 < 1 - 2c < 1$$

$\Rightarrow$ $0 < c < 1$

Because $c > 0$ (A, Δt, Δx are all positive)

the condition for stability is

$$\boxed{c < 1}$$

(c) With the modified stability scheme, we ~~got~~ have:

$$u_i^{n+1} = u_i^n - c\,(u_{i+1}^n - u_i^n)$$

Let $u_i^n = \epsilon$ $u_{i+1}^n = -\epsilon$

$\Rightarrow$ $u_i^{n+1} = \epsilon - c(-\epsilon - \epsilon)$

or $$\frac{u_i^{n+1}}{\epsilon} = 1 + 2c$$

For stability

$$-1 < 1 + 2c < 1$$

As $c > 0$, this condition cannot be satisfied for any value of c.

The reason is that the governing equation is hyperbolic with a right-travelling wave. We need a backward ∂ difference scheme (the first scheme) to be within the domain of dependence. The modified scheme is a forward difference scheme, so is unstable.

This was a very popular and well answered question, possibly a bit too easy.

Q7

a)

Hyperbolic.

Wave equation: $\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$

Initial value problem. Information propagates along characteristics. If the wave is travelling to the right, use a backward differencing ~~sch~~ scheme or 'upwinding'.

Elliptic

$\nabla^2 \phi = 0$ Potential flow

Boundary value problem. Every point in the domain affects every other point. As there are no characteristics, central difference schemes work well.

Parabolic

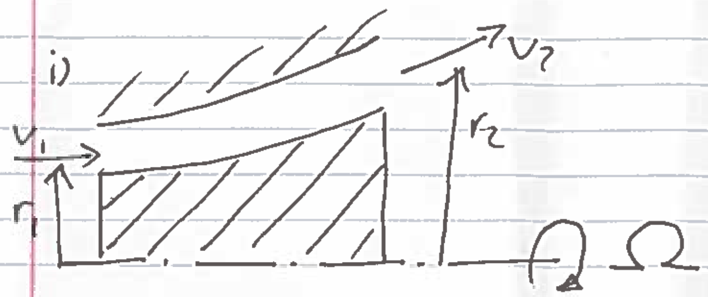
Heat conduction $\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} = 0$.

Initial & boundary value problem. Signal travel in a particular direction and diffuse in space. ~~in time~~

Use a time marching scheme, that is central difference in space.

Q7 cont.

b) i)



Assume steady flow

Change in angular momentum = torque

$$\text{Torque} = \dot{m}\left(r_2 V_{\theta_2} - r_1 V_{\theta_1}\right)$$

V_θ: flow in tangential direction

Power: $\dot{W}_x = \text{Torque} \cdot \Omega$

$$= \dot{m}\,\Omega\left(r_2 V_{\theta_2} - r_1 V_{\theta_1}\right)$$

Note: $\Omega r = U$

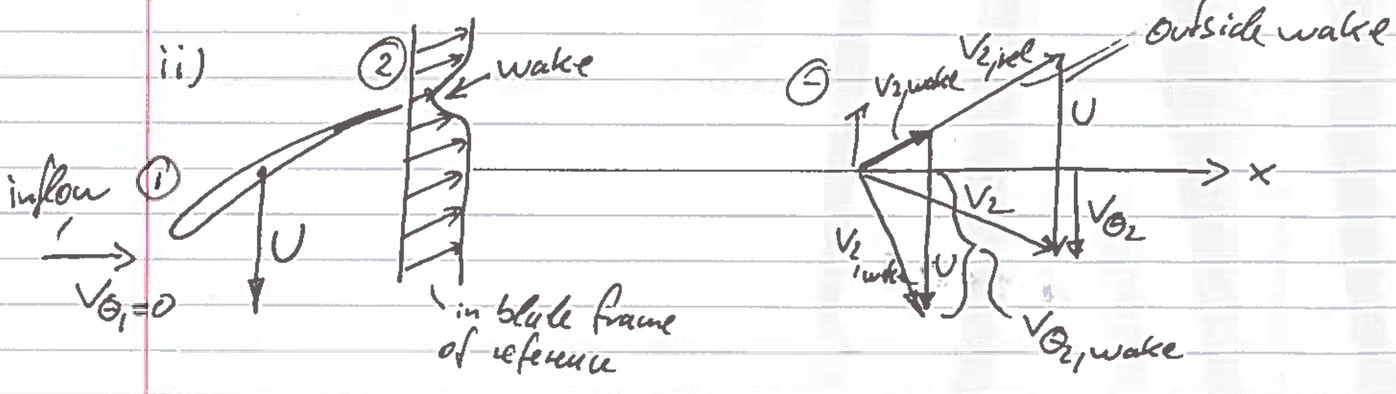
$$\therefore \quad \frac{\dot{W}_x}{\dot{m}} = U_2 V_{\theta_2} - U_1 V_{\theta_1}$$

Steady flow energy equation: $\dot{Q} - \dot{W}_x = \Delta h_0 \, \dot{m}$

thus, assuming adiabatic flow

$$\frac{\dot{W}_x}{\dot{m}} = \Delta\left(U V_\theta\right) \qquad \text{q.e.d.}$$

ii)



Euler work eqn.: $\Delta h_0 = \Delta\left(U V_\theta\right) = U V_{\theta_2} \quad (V_{\theta_1} = 0)$

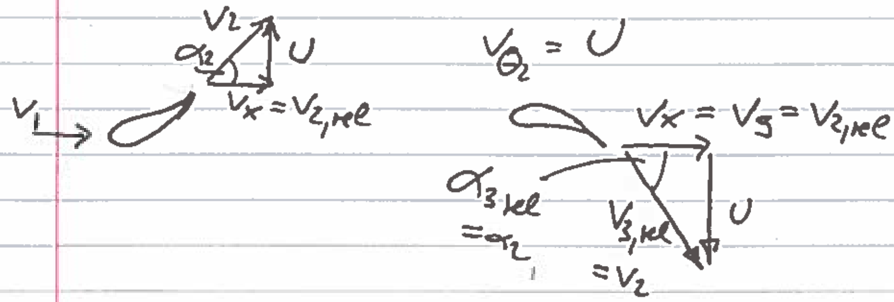
from sketch: $V_{\theta_2,wake} > V_{\theta_2}$

$$\therefore \quad h_{02,wake} > h_{02}$$

$$\therefore \quad T_{02,wake} > T_{02} \qquad \text{q.e.d.}$$

Q 8

a) <u>Reaction stage</u> $\Lambda = 50\%$

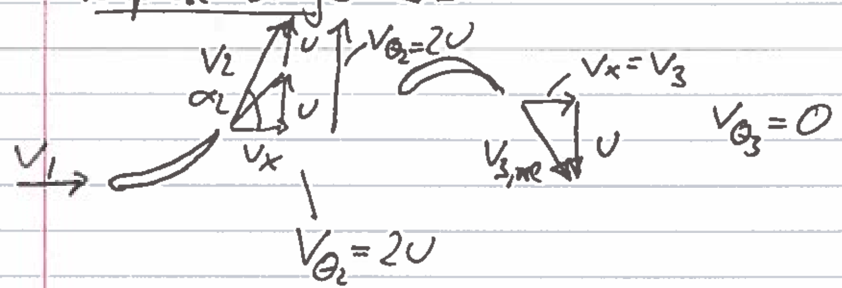


Note similarity in velocity triangles, thus $V_{3,rel} = V_2$

$$\Delta h_0 = \Delta(U V_\theta) = U(U-0) = U^2$$

$$\therefore \psi = \frac{\Delta h_0}{U^2} = 1 \qquad \text{(stage loading coefficient)}$$

<u>Impulse stage</u> $\Lambda = 0\%$



$$\Delta h_0 = U(2U-0) = 2U^2$$

$$\therefore \psi = \frac{\Delta h_0}{U^2} = 2$$

b) For each row, $Y = \dfrac{p_{01} - p_{02}}{p_{02} - p_2} = \dfrac{\Delta p_0}{\frac{1}{2}\rho V_2^2}$

Reaction stage : $Y_R = \dfrac{\Delta p_{0R}}{\frac{1}{2}\rho V_2^2}$ (same Δp_0 for rotor and stator)

From velocity triangle : $\cos\alpha_2 = \dfrac{V_x}{V_2}$ $\quad \tan\alpha_2 = \dfrac{U}{V_x} = \dfrac{1}{\phi}$

thus : $\phi = \dfrac{V_x}{U} = \dfrac{\cos\alpha_2}{\sin\alpha_2} = \dfrac{\cos\alpha_2}{\sqrt{1-\cos^2\alpha_2}} = \dfrac{\frac{V_x}{V_2}}{\sqrt{1-\left(\frac{V_x}{V_2}\right)^2}}$

$\therefore \left(\dfrac{V_x}{V_2}\right)^2 (1+\phi^2) = \phi^2$ $\quad$ Thus : $\dfrac{\Delta p_{0R}}{\frac{1}{2}\rho V_x^2} = Y_R \cdot \dfrac{1+\phi^2}{\phi^2}$

Total loss : $2 \cdot Y_R \dfrac{1+\phi^2}{\phi^2}$ for stage

Q8 cont.)

b) cont. Impulse stage $Y_I = \frac{\Delta p_{0I}}{\frac{1}{2}\rho V_2^2}$ (for rotor + stator)

but $\tan\alpha_2 = \frac{2U}{V_x}$ (stator)

and $\tan\alpha_3 = \frac{U}{V_x}$ (rotor)

Thus $\left.\frac{\Delta p_{0I}}{\frac{1}{2}\rho V_x^2}\right|_{\text{stator}} = Y_I \frac{4+\phi^2}{\phi^2}$ (derivation as before)

and $\left.\frac{\Delta p_{0I}}{\frac{1}{2}\rho V_x^2}\right|_{\text{rotor}} = Y_I \frac{1+\phi^2}{\phi^2}$

Loss for stage $\left.\frac{\Delta p_{0I}}{\frac{1}{2}\rho V_x^2}\right|_{\text{impulse stage}} = Y_I\left[\frac{4+\phi^2}{\phi^2} + \frac{1+\phi^2}{\phi^2}\right]$

$$= Y_I \frac{5+2\phi^2}{\phi^2}$$

c) Impulse stage more highly loaded → thus likely to have greater loss

But: Need two reaction stages for same power output.

Loss for two reaction stages: $4 Y_R \frac{1+\phi^2}{\phi^2}$

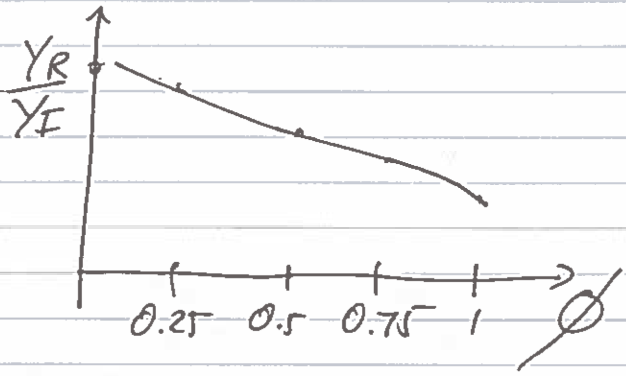
for one impulse stage: $Y_I \frac{5+2\phi^2}{\phi^2}$

For break-even $Y_R(4+4\phi^2) = (5+2\phi^2) Y_I \quad \therefore \quad \frac{Y_R}{Y_I} = \frac{5+2\phi^2}{4+4\phi^2}$

ϕ	1	0.75	0.5	0.25
$\frac{Y_R}{Y_I}$ break-even	0.875	0.98	1.10	1.21

Q8 cont.

c) cont.



Loss levels increase as ϕ reduces. For same overall loss the reaction stage must have significantly lower loss levels as ϕ increases. Reaction stage benefits from lower loading which should reduces losses. At low ϕ (undesireable) the reaction stage can have comparable losses to impulse stage to break even. → in practice the are likely to be lower so reaction design will be more efficient there. Conversely at high ϕ impulse design may eventually win over.

Many candidates failed to see the significance of the reaction percentage on the velocity triangles