Question 1 Mass diffusion through sensor

Solution (a)(i) In steady state, the diffusion flux across the layer must be equal to the convection rate and to the reaction:

$$j = h_m(c_b - c_s) = \frac{D}{\delta}(c_s - c_c) = \Omega = Ac_c$$

$$\frac{c_b - c_s}{1/h_m} = \frac{c_s - c_c}{\delta/D} = Ac_c$$

$$c_b - c_s = \Omega/h_m = Ac_c/h_m$$

$$c_s - c_c = \Omega\delta/D = Ac_c\delta/D$$

$$c_b - c_c = Ac_c\left(\frac{1}{h_m} + \frac{\delta}{D}\right)$$

$$c_c = c_b\frac{1}{1 + \dfrac{A}{h_m} + \dfrac{A\delta}{D}}$$

$$c_s = c_b - \frac{A}{h_m}c_c = c_b\left(1 - \frac{A}{h_m}\frac{1}{1 + \dfrac{A}{h_m} + \dfrac{A\delta}{D}}\right) = c_b\frac{1 + \dfrac{A}{h_m} + \dfrac{A\delta}{D} - \dfrac{A}{h_m}}{1 + \dfrac{A}{h_m} + \dfrac{A\delta}{D}}$$

$$c_s = c_b\frac{1/A + \delta/D}{1/A + 1/h_m + \delta/D}$$

The main problem students encountered in setting up this problem is recognising the difference between a *flux* in steady state and a *rate* in the form of an unsteady term. Keep in mind that this is the same as a steady state heat transfer problem, so it can be handled via the circuit analogy. The second thing to keep in mind is that, although we do not know yet what the concentrations are, we are looking for relationships between the concentrations – in this case there are two, at the two interfaces, so that one can eliminate the intermediate value of c_s as an unknown.

(a)(ii) We can now use this expression to determine the flux:

$$j = Ac_c = c_b\frac{A}{1 + A(1/h_m + \delta/D)} = c_b\frac{1}{1/A + 1/h_m + \delta/D}$$

When A is much smaller than the diffusion and convection terms, we have:

$$c_c \approx c_s \approx c_b \qquad j = Ac_b$$

and the overall flux is controlled by the rate of reaction.

When A is much larger than the diffusion and convection terms, we have:

$$c_c \to 0 \qquad c_s = c_b\frac{\delta/D}{1/h_m + \delta/D} \qquad j = \frac{D}{\delta}c_s = c_b\frac{1}{1/h_m + \delta/D}$$

and the overall flux is controlled by the rate of diffusion.

(b)(i) The response time is given by the diffusion time through the layer of thickness δ:

$$\tau = \delta^2/D$$

This is the easiest part of the problem! Diffusion times can always be estimated in this manner, to the order of magnitude.

(b)(i)

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$$

Substituting the given equation, we have:

$$\frac{\partial}{\partial t}[C_n \exp(-k_n^2 t/\tau) \sin(k_n x/\delta)] = C_n \sin(k_n x/\delta)(-k_n^2)\frac{D}{\delta^2} \exp(-k_n^2 \tau)$$
$$\frac{\partial}{\partial x}[C_n \exp(-k_n^2 t/\tau) \sin(k_n x/\delta)] = C_n (k_n/\delta) \exp(-k_n^2 \tau) \cos(k_n x/\delta)$$
$$\frac{\partial^2}{\partial x^2}[C_n \exp(-k_n^2 t/\tau) \sin(k_n x/\delta)] = C_n(-\frac{k_n^2}{\delta^2}) \exp(-k_n^2 \tau) \sin(k_n x/\delta)$$

so that, for each term:

$$C_n \sin(k_n x)(-k_n^2)\frac{D}{\delta^2} \exp(-k_n^2 t/\tau) = DC_n(-\frac{k_n^2}{\delta^2}) \exp(-k_n^2 \tau) \sin(k_n x/\delta)$$

Clearly, the solution agrees with the necessary equation.

(b)(ii) The boundary condition at $x = 0$ is:

$$C_n \exp(-k_n^2 t/\tau) \sin(k_n 0) = 0$$

is satisfied automatically, and that at $x = \delta$

$$C_n \exp(-k_n^2 t/\tau) \sin(k_n) = c_{s1}$$

can be satisfied by taking $k_n = \pi(n + 1/2)$ for a non-zero value, the values of C_n can be worked out via a Fourier transform of the step function.

Question 2 Heat exchanger

A simple progressive question, that tripped up students, as the geometrical setup was perhaps a little difficult to explain. Most students worked out the log mean temperature and the relationship with heat flux. Problems appeared in accounting correctly for the flux through the fins, which required a simple multiplication by the appropriate area and efficiency. The last part was duly challenging and was only attempted by a few.

Solution (a) The heat transfer from air at a temperature T to a surface at T_s across a resistance U over an area $W\ dx$ is:

$$dq_o = UW(T - T_s)\ dx = -\dot{m}c_p dT = -\dot{m}(T - T_s)$$
$$\frac{d(T - T_s)}{T - T_s} = -\frac{UWL}{\dot{m}c_p}\ dx$$
$$\ln\frac{(T_{h,o} - T_s)}{(T_{h,i} - T_s)} = -\frac{UWL}{\dot{m}c_p}$$

$$q_o = \dot{m}c_p(T_{h,i} - T_{h,o}) = -\frac{UWL(T_{h,i} - T_{h,o})}{\ln\dfrac{(T_{h,o} - T_s)}{(T_{h,i} - T_s)}} = UWL\theta_l$$

where the resistance is:

$$1/U = 1/h + (\delta/2)/\lambda$$

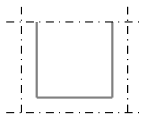

Figure 1: unit element

(b) Considering the unit symmetry element in Fig. 1, we can see that the total heat transferred from the air to the surface is the sum of the heat directly transferred to the surface of the heat exchanger over the area $(W - \delta)L$, plus the heat transfer through the fins over area δL. The latter can be considered as the heat transfer over δL, enhanced by the effectiveness η.

$$q = q_{base} + q_{fin} = q_o\frac{(W - \delta)L}{WL} + q_o\eta\frac{\delta L}{WL} = q_o[1 - \frac{\delta}{W} + \frac{\delta}{W}\eta] = q_o[1 + \frac{\delta}{W}(\eta - 1)]$$

The length of the fins (symmetric about top and bottom) is $L_f = H/2$, the cross sectional area of the (half) fins, $A_c/2 = \delta L/2$, so that the total cross sectional area of the fin is $A_c = \delta L$. The wetted perimeter of the fins is $P_f = 2L$ (both sides). The fin efficiency can be calculated as:

$$m = \sqrt{hP_f/(\lambda A_c)} = \sqrt{h2L/(\lambda\delta L)} = \sqrt{2h/\lambda\delta}$$
$$\eta = \tanh(\sqrt{2h/\lambda}H/2)/(\sqrt{2h/\lambda\delta}H/2)$$

The total heat transfer rate in one unit is therefore:

$$q_u = q_o[1 + \frac{\delta}{W}(\eta - 1)] = UWL\theta_l[1 + \frac{\delta}{W}(\eta - 1)]$$

where the parameters can be substituted above.

The typical confusion in the solutions related to how to treat the fins relatively to the surface. In general, marking was generous, so that small deviations in interpretation were considered acceptable answers.

(c)

For N channels, the heat transfer is N times q_u for each unit, times 2 for each of the ends of the stack:

$$Q = 2Nq_u$$

(d)

We consider the heat exchange for a unit stack of width and length W, but now between the hot and cold gases.

$$\Delta q_u = -\dot{m}c_p\Delta T_h = +\dot{m}c_p\Delta T_c = U'W^2[1 + \frac{\delta}{W}(\eta - 1)]U'(T_h - T_c)$$

Since the heat capacities of each stream are identical, we have $\Delta T_h = -\Delta T_c$. The heat transfer coefficient is given by:

$$\frac{1}{U'} = \frac{2}{h} + 2\frac{\delta/2}{\lambda} = \frac{2}{U}$$

$$U' = \frac{U}{2}$$

We can now solve for the evolution in the temperature difference, which will be the same in the two directions.

$$\Delta(T_h - T_c) = -U'W^2[1 + \frac{\delta}{W}(\eta - 1)](T_h - T_c)$$

If the initial conditions are known, the equation can be integrated at the limit of small W^2 as a differential in the two directions.

Question 3 Hot film sensor

Boring but popular question. The first part on the calculation of the mean Nusselt number tripped many students (all but one), even though the definition was given. The second part involved calculating numerical values for Re and Nu, which was straight forward, and the third item required sketching the momentum and thermal boundary layers. About half of the students got it right, the other half got it backwards or did not recognize the role of Pr. The final item required an evaluation of the assumptions for the flat plate, which few attempted, but sensible answers were also given.

Solution

(a)

The heat transfer coefficient $h_x = \lambda Nu_x/x = \lambda C Re_x^m/x = \lambda C(U/\nu)^m x^m/x$. The mean heat transfer coefficient over a length L is:

$$\overline{h}_L = \frac{1}{L}\int_0^L \lambda C(U/\nu)^m x^{m-1}\,dx = \frac{\lambda C(U/\nu)^m L^m}{mL} = \frac{\lambda Nu_L}{m}$$

$$\overline{Nu_L} = \overline{h}_L L/\lambda = Nu_L/m$$

For the laminar case,

$$\overline{Nu_L} = (2)0.332 Pr^{1/3} Re_L^{1/2}$$

For the turbulent case,

$$\overline{Nu_L} = \frac{5}{4}0.0385 Pr^{1/3} Re_L^{4/5}$$

If the transition to turbulence takes place at ℓ, we must calculate the convection coefficient accordingly:

$$\begin{aligned}\overline{h}_L &= \int_0^\ell \lambda C(U/\nu)^m x^{m-1}\,dx + \int_\ell^L \lambda C(U/\nu)^{m'} x^{m'-1}\,dx \\ &= \frac{\lambda Nu_{\ell,l}}{m} + \frac{\lambda Nu_{L,t}}{m'} - \frac{\lambda Nu_{\ell,t}}{m'} \\ \overline{Nu_L} &= \frac{\overline{h}_L L}{\lambda} = \frac{Nu_{\ell,l}}{m} + \frac{Nu_{L,t}}{m'} - \frac{Nu_{\ell,t}}{m'}\end{aligned}$$

(b)

Property	Air	Water
μ (kg/m·s)	17.9 $\times 10^{-6}$	1.15 $\times 10^{-3}$
ρ (kg/m^3)	1.22	1000
ν (m^2/s)	1.47$\times 10^{-5}$	1.15 $\times 10^{-6}$
Pr (-)	0.70	8.12

which yields the following results

	Air		Water	
Case	A	B	C	D
U (m/s)	1	10	1	10
Re_L (-)	34	3401	435	43478
Nu_L	1.7	17.2	13.9	397 (t)
$Nu_{\ell,t}$				123
$Nu_{\ell,l}$				66.7
$\overline{Nu_L}$	3.4	34	28	477

Most inaccuracies in the exam came from the reasonable assumption that the mean Nu can be calculated from the direct integration. However, because of the definition of Nu and its dependence on distance, this is somewhat different than the final result here. Nevertheless, most of the credit was given if the general understanding was correct regarding the averaging of Nu when there is a transition.

(b) We have:

$$\begin{aligned} q = \overline{L}(T_h - T_\infty) = \lambda \overline{Nu}_L / L = CU^m \\ \frac{\Delta q}{q} = m \frac{\Delta U}{U} \\ \frac{\Delta U}{U} = \frac{1}{m} \frac{\Delta q}{q} \end{aligned}$$

where $m = 1/2$ for laminar and $4/5$ for turbulent flow. Therefore, the sensitivity is larger for laminar than turbulent flow, and so is the uncertainty multiplier.

(c) The sketch should have the appropriate labelling of the gradients at wall and zero gradient at the boundary layer, the thickness of the boundary layer, and the relative magnitudes. In particular, we note that the momentum and thermal boundary layer thickness *decrease* with *increasing* Reynolds and Nusselt number, and that the momentum boundary layer is *thicker* than the thermal boundary layer when Prandlt number $Pr = \nu/\alpha$ is larger than unity. The figure below is illustrative.

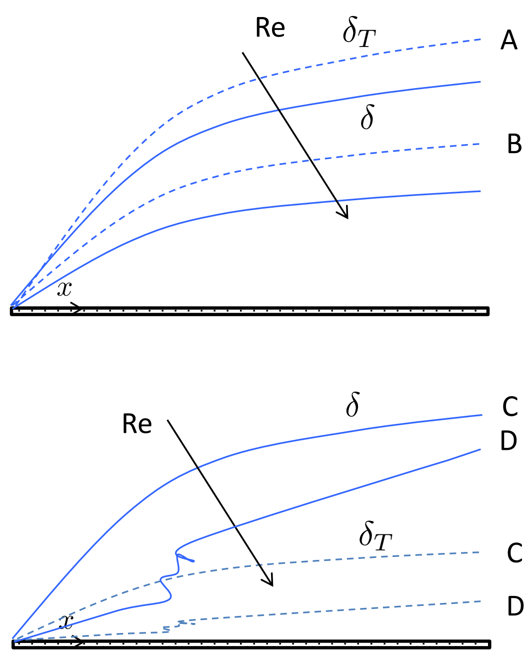

Figure 2:

(d) The derivations use the assumption of a thin boundary layer relative to the length of the system, which clearly fails a the leading edge, and in the present case, also at the trailing edge for the lowest velocity case, since the boundary layer thickness for 1 m/s for air is 1 mm, for a sensor 5 mm in length. However, for the other cases the estimate shows that the boundary layer is indeed very thin and that the analysis holds up. Ways around the limitation are: (i) full numerical calculations, (ii) calibration, (iii) measurement of drag and use heat and momentum transfer analogy.

Question 4 Quarter cylinder slab

Most attempts successfully derived the appropriate differential equation from scratch, and went on to integrate and apply the appropriate boundary conditions in the second part. Although most students did discretise and analyse how to solve it, in most cases there was a certain level of sloppiness in taking into account the changing radius. However, much of which was given credit if the overall picture was adequately addressed.

(a) We consider the steady energy equation for a differential element $dr \times r\ d\theta \times dz$:

$$\begin{aligned}
\frac{\partial E}{\partial t} = \quad & q_r r\ d\theta\ dz - q_{r+dr}(r+dr)\ d\theta\ dz + q_\theta\ dr\ dz - q_{\theta+d\theta}\ dr\ dz + \\
& + q_z\ dr\ r\ d\theta - q_{z+dz}\ dr\ r\ d\theta + \dot{g}\ dr\ r\ d\theta dz = 0 \\
& q_r r\ d\theta\ dz - [q_r r + \frac{\partial}{\partial r}(r q_r) dr]\ d\theta\ dz + q_\theta\ dr\ dz - [q_\theta + \frac{\partial q_\theta}{\partial \theta} d\theta]\ dr\ dz \\
& + q_z\ dr\ r\ d\theta - [q_z + \frac{\partial q_z}{\partial z} dz]\ dr\ r\ d\theta + \dot{g}\ dr\ r\ d\theta dz = 0 \\
& \frac{1}{r}\frac{\partial}{\partial r}(q_r r) + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} - \dot{g} = 0
\end{aligned}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(-\lambda r \frac{\partial T}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(-\frac{\lambda}{r}\frac{\partial T}{\partial \theta}) + \frac{\partial}{\partial z}(-\lambda \frac{\partial T}{\partial z}) - \dot{g} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{\lambda} = 0$$

(b) We have a solution that depends only on r, so that:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) &= -\frac{\dot{g}}{\lambda} \\
\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) &= -\frac{\dot{g}}{\lambda} r \\
r\frac{\partial T}{\partial r} &= -\frac{\dot{g}}{\lambda}\frac{r^2}{2} + A \\
\frac{\partial T}{\partial r} &= -\frac{\dot{g}}{\lambda}\frac{r}{2} + \frac{A}{r} \\
T = -\frac{\dot{g}}{\lambda}\frac{r^2}{4} &+ A \ln r + B
\end{aligned}$$

Applying the boundary conditions of symmetry at r and convection at R, we have:

$$\begin{aligned}
A &= 0 \\
-\lambda \frac{\partial T}{\partial r}\Big|_{r=R} = \dot{g}\frac{R}{2} = h(T(R) - T_\infty) T(R) &= \dot{g}\frac{R}{2h} + T_\infty \\
T(0) = T(R) + \frac{\dot{g}}{\lambda}\frac{R^2}{4} &= B \\
T = T_\infty + \dot{g}\frac{R}{2h} + \frac{\dot{g}}{\lambda}\frac{R^2}{4}&(1 - \frac{r^2}{R^2})
\end{aligned}$$

(c) Now we consider different conditions, and azimuthal symmetry no longer applies. (i) Assuming a separable solution, we have $T = R(r)\Theta(\theta)$. The differential equation can be expressed as :

$$\Theta\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + R\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} + \frac{\dot{g}}{\lambda} = 0$$

The possible solutions are series solutions, starting from the homogeneous solutions, and using these to obtain a particular solution.

(d) We seek a discretization of the differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{\dot{g}}{\lambda} = 0$$

with the gradient specified at the boundary. We specify cell enters at points i, j corresponding to r and θ. The discretisation of the gradients reads, for a first order differencing scheme:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \approx & \frac{1}{r_i}\frac{1}{\Delta r}\left[r_{i+1/2,j}\frac{\partial T}{\partial r}\bigg|_{i+1/2,j} - r_{i-1/2,j}\frac{\partial T}{\partial r}\bigg|_{i-1/2,j}\right] = \\
& \frac{1}{r_{i,j}}\frac{1}{\Delta r}\left[\frac{r_{i+1,j}+r_{i,j}}{2}\frac{T_{i+1,j}-T_{i,j}}{\Delta r} - \frac{r_{i,j}+r_{i-1,j}}{2}\frac{T_{i,j}-T_{i-1,j}}{\Delta r}\right] = \\
& \frac{1}{2}\frac{1}{\Delta r^2}\left[(\frac{r_{i+1,j}}{r_{i,j}}+1)T_{i+1,j} - (\frac{r_{i+1,j}}{r_{i,j}} + \frac{r_{i-1,j}}{r_{i,j}} + 2)T_{i,j} + (\frac{r_{i-1,j}}{r_{i,j}}+1)T_{i-1,j}\right] \\
\frac{1}{r^2}\frac{\partial T}{\partial\theta} \approx & \frac{1}{r_i^2}\frac{1}{\Delta\theta^2}\left[\frac{\partial T}{\partial\theta}\bigg|_{i,j+1/2} - \frac{\partial T}{\partial\theta}\bigg|_{i,j-1/2}\right] = \\
& \frac{1}{r_{i,j}^2}\frac{1}{\Delta\theta^2}\left[T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right]
\end{aligned}$$

For the internal points, we have therefore, after collecting terms:

$$\begin{aligned}
& \frac{1}{2}\frac{1}{\Delta r^2}\left[(\frac{r_{i+1,j}}{r_{i,j}}+1)T_{i+1,j} - (\frac{r_{i+1,j}}{r_{i,j}} + \frac{r_{i-1,j}}{r_{i,j}} + 2)T_{i,j} + (\frac{r_{i-1,j}}{r_{i,j}}+1)T_{i-1,j}\right] + \\
& \qquad \frac{1}{r_{i,j}^2}\frac{1}{\Delta\theta^2}\left[T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right] = 0 \\
& \frac{1}{2}\frac{1}{\Delta r^2}(\frac{r_{i-1,j}}{r_{i,j}}+1)T_{i-1,j} - [\frac{1}{2}\frac{1}{\Delta r^2}(\frac{r_{i+1,j}}{r_{i,j}} + \frac{r_{i-1,j}}{r_{i,j}} + 2) + \frac{2}{r_{i,j}^2}\frac{1}{\Delta\theta^2}]T_{i,j} + \\
& \qquad \frac{1}{2}\frac{1}{\Delta r^2}(\frac{r_{i+1,j}}{r_{i,j}}+1)T_{i+1,j} + \frac{1}{r_{i,j}^2}\frac{1}{\Delta\theta^2}T_{i,j-1} + \frac{1}{r_{i,j}^2}\frac{1}{\Delta\theta^2}T_{i,j+1} = 0
\end{aligned}$$

For the boundary points, we can discretise using forward differentiation for the gradient condition at $\theta = 0 (j = 1)$ and $\theta = \pi/2 (j = M)$:

$$\begin{aligned}
-\frac{\lambda}{r}\frac{\partial T}{\partial\theta} &= h(T - T_\infty) \\
-\frac{\lambda}{r_{i,1}}\frac{T_{i,2}-T_{i,1}}{\Delta\theta} &= h(T_{i,1} - T_\infty)
\end{aligned}$$

$$\begin{aligned}
-\frac{\lambda}{r}\frac{\partial T}{\partial\theta} &= h(T - T_\infty) \\
-\frac{\lambda}{r_{i,N}}\frac{T_{i,M}-T_{i,M-1}}{\Delta\theta} &= h(T_{i,M} - T_\infty)
\end{aligned}$$

And for the circular boundary $i = N$:

$$-\lambda \frac{\partial T}{\partial r} = h(T - T_\infty)$$
$$-\lambda \frac{T_{N-1,j} - T_{N,1}}{\Delta r} = h(T_{N,j} - T_\infty)$$

Each one of the discretized equations defines a linear system involving the temperatures at each node, such that $\mathbf{AT} = \mathbf{B}$, which can be solved by standard methods.