Q1. NAT. CONV. 1/①

(a) $p_c(y) = p_{c0} - \rho_c g y$ HYDROSTATIC PRESSURE

$p_h(y) = p_{h0} - \rho_h g h$

$$(p_c - p_h) = (p_{c0} - p_{h0}) - (\rho_c - \rho_h) g y$$

EQUAL MEAN PRESSURE: $\frac{1}{H}\int (p_c - p_h) = 0$

$$(p_{c0} - p_{h0}) - (\rho_c - \rho_h)\frac{gH}{2} = 0$$

$$\Delta p = (p_{c0} - p_{h0}) = (\rho_c - \rho_h)\frac{gH}{2} \quad ; \quad (p_{cH} - p_{hH}) = -(\rho_c - \rho_h)\frac{gH}{2}$$

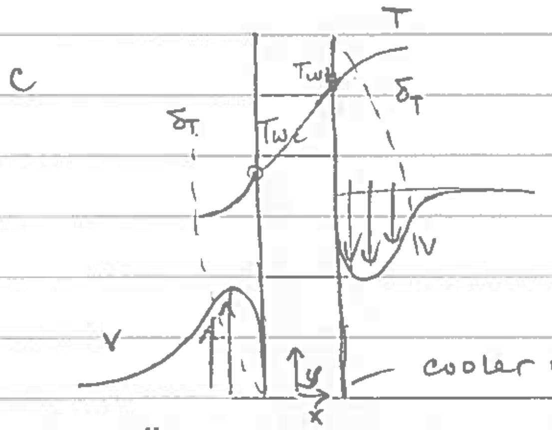
—"—

$$u(y) = \frac{D^2}{8\mu}\frac{\Delta p}{L}\left[1 - \left(\frac{2y}{D}\right)^2\right]$$

$$\dot{m} = 2W\rho\int_0^{D/2} u\,dy = \frac{2\rho W \Delta p}{8\mu L}\left[\frac{D}{2} - \frac{4}{D^2}\frac{(D/2)^3}{D^2}\right] = \frac{1}{12}\frac{\Delta p}{\nu L} D^3 W$$

$$\left[\dot{m} = \frac{1}{12}\frac{D^3 W}{\nu L}(\rho_c - \rho_h)\frac{gH}{2} = \frac{1}{24}\frac{D^3 W H g}{\nu L}(\rho_c - \rho_h)\right]$$

(b) (i)



Hotter wall – flows up

(NOTE: y and x reversed from usual boundary layer orientation)

(ii) mass $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

$$\frac{u}{\delta_T} \sim \frac{v}{H_T} \qquad \left[v \sim u \frac{\delta_T}{H} \right]$$

ENERGY: $\quad u \dfrac{\partial T}{\partial x} + v \dfrac{\partial T}{\partial y} = \alpha \dfrac{\partial^2 T}{\partial x^2}$

$$v \frac{\delta_T}{H} \frac{\Delta T}{\delta_T} \sim v \frac{\Delta T}{H} \sim \frac{\alpha \Delta T}{\delta_T^2}$$

$$\left[v \sim \frac{\alpha H}{\delta_T^2} \right]$$

(iii) vert. mom: $\quad u \dfrac{\partial v}{\partial x} + v \dfrac{\partial v}{\partial y} = \nu \dfrac{\partial^2 v}{\partial x^2} + g\beta (T - T_\infty)$

$$v \frac{\delta_T}{H} \frac{v}{\delta_T} \quad \frac{v^2}{H} \quad \underbrace{\nu \frac{v}{\delta_T^2} \sim g\beta \Delta T}_{\text{LEADING ORDER}}$$

$$\nu \left(\frac{\alpha H}{\delta_T^2} \right) \frac{1}{\delta_T^2} \sim g\beta \Delta T$$

$$\delta_T^4 \sim \frac{\alpha \nu H}{g\beta \Delta T} \longrightarrow \left[\left(\frac{\delta_T}{H} \right) = \underbrace{\left[\frac{\alpha \nu}{g\beta \Delta T H^3} \right]^{1/4}}_{Ra_H^{-1/4}} \right]$$

✓

(c) MASS FLOW RATE ASSOCIATED WITH N.CONV.

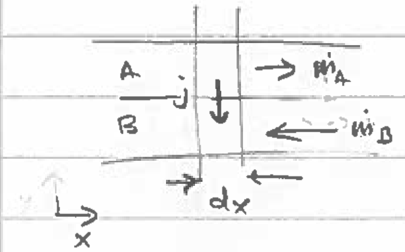
$$\dot{m}_{NC} \sim \rho v \delta_H W \sim \rho \frac{\alpha H W}{\delta_T}$$

$$\sim \rho \frac{\alpha \cancel{H} W}{\cancel{H}} \left(\frac{g \beta \Delta T}{\alpha \nu H^3} \right)^{1/4}$$

$$\frac{\dot{m}_{NC}}{\dot{m}_L} = \left(\frac{\rho}{\Delta\rho}\right) \alpha \left(\frac{g \beta \Delta T}{\alpha \nu H^3} \right)^{1/4} \frac{24 \nu L}{g D^3 H} \qquad \frac{\Delta \rho}{\rho} \sim \beta \Delta T$$

$$= 24 \left(\frac{\alpha \nu}{g \beta \Delta T D^3} \right) \left(\frac{L}{H} \right) \left(\frac{g \beta \Delta T}{\alpha \nu H^3} \right)^{1/4}$$

$$\boxed{\frac{\dot{m}_{NC}}{\dot{m}_L} = 24 \underbrace{\left(\frac{\alpha \nu}{g \beta \Delta T H^3} \right)^{+3/4}}_{Ra_H^{-1}} \left(\frac{H}{D} \right)^3 \left(\frac{L}{H} \right)}$$



(a) Ⓐ $-\dot m_A\left(Y_A + \frac{dY_A}{dx}dx\right) + \dot m_A Y_A - j = 0$

$$\dot m_A \frac{dY_A}{dx} = -j$$

Ⓑ $+\dot m_B\left(Y_B + \frac{dY_B}{dx}dx\right) - \dot m_B Y_B + j = 0$

$$\dot m_B \frac{dY_B}{dx} = -j$$

$j = \rho h_A (Y_A - Y_w)$
$j = \rho h_B (Y_A - Y_w)$

$$j = \frac{\rho}{1/h_A + 1/h_B}(Y_A - Y_B) = \rho U (Y_A - Y_B)$$

(b) $$\int_0^L j\,dx = \int_0^L \rho U (Y_A - Y_B)\,dx$$

$$\frac{dY_A}{dx} = -\frac{j}{\dot m_A} \qquad \frac{d(Y_A - Y_B)}{dx} = -j\left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right)$$

$$\frac{dY_B}{dx} = -\frac{j}{\dot m_B}$$

$$\int_0^L -\frac{d(Y_A - Y_B)}{1/\dot m_A - 1/\dot m_B} = \int_0^L j\,dx = J$$

$$(Y_A - Y_B)_0 - (Y_A - Y_B)_L = \left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right) J$$

$$\frac{d(Y_A - Y_B)}{dx} = -j\left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right) = -u\left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right)(Y_A - Y_B)$$

$$d\ln(Y_A - Y_B) = -\rho u\left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right) dx$$

$$\ln \frac{\Delta Y_L}{\Delta Y_0} = -\rho u L\left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right)$$

$$\boxed{J = \rho \frac{\Delta Y_0 - \Delta Y_L}{1/\dot m_A - 1/\dot m_B} = \rho u L \frac{\Delta Y_0 - \Delta Y_L}{\ln \Delta Y_0/\Delta Y_L}}$$

(c) $$\ln\left[\frac{(Y_A - Y_B) - (Y_{A0} - Y_{B0})}{(Y_A - Y_B)_L - (Y_A - Y_B)_0}\right] = -\rho u\left(\frac{1}{\dot m_A} - \frac{1}{\dot m_B}\right) x$$

OVERALL BALANCE:

$$\dot m_A(Y_{A0} - Y_{AL}) = \dot m_B(Y_{B0} - Y_{BL})$$

$$\dot m_A(0.5\,Y_{A0}) = \dot m_B Y_{B0} \longrightarrow Y_{B0} = 0.5\frac{\dot m_A}{\dot m_B} Y_{A0}$$

$$\Delta Y_L = Y_{AL} - Y_{BL} = Y_{AL} = 0.5\,Y_{A0}$$

$$\Delta Y_0 = Y_{A0} - Y_{B0} = Y_{A0} - 0.5\frac{\dot m_A}{\dot m_B} Y_{A0} = (1 - 0.5\beta) Y_{A0}$$

$$(Y_A - Y_B) = (1 - 0.5\beta) Y_{A0} - \left(0.5\,Y_{A0} - (1 - 0.5\beta) Y_{A0}\right) \cdot$$
$$\cdot \exp\left(\frac{-\rho u}{\dot m_A}(1-\beta) x\right)$$

$$J = \rho u L \frac{\left[(1 - 0.5\beta) - 0.5\beta\right]}{\ln\left[\frac{1 - 0.5\beta}{0.5}\right]} c_{A0} = \rho u L \left[\frac{1-\beta}{\ln[2-\beta]}\right]\frac{1}{2} Y_{A0}$$

(d) $\ln\left[\frac{\chi_A-\chi_B-(\chi_{A0}-\chi_{B0})}{(\chi_A-\chi_B)_L-(\chi_A-\chi_B)_0}\right] = -\frac{\rho u}{\dot{m}_A}\left(1-\beta\right)x$

$$\ln\left[\frac{(\chi_A-\chi_B)-(1-0.5\beta)\chi_{A0}}{0.5\chi_{A0}-(1-0.5\beta)\chi_{A0}}\right] = -\frac{\rho u}{\dot{m}_A}(1-\beta)x$$

$$-(1-0.5\beta)\chi_{A0}+(\chi_A-\chi_B) = \left[0.5(-1+\beta)\chi_{A0}\right]\exp\left(\frac{-\rho u}{\dot{m}_A}(1-\beta)x\right)$$

$$\chi_A-\chi_B = (1-0.5\beta)\chi_{A0}+0.5(-1+\beta)\chi_{A0}\exp\left(\frac{-\rho u}{\dot{m}_A}(1-\beta)x\right)$$

check: $(\chi_A-\chi_B)(0) = \left[(1-0.5\beta)+0.5(-1+\beta)\right]\chi_{A0} = 0.5\chi_{A0}$ ✓

$$j = u\chi_{A0}\left[\underbrace{(1-0.5\beta)}_{A}-\underbrace{0.5(1-\beta)}_{B}\exp\left(\underbrace{\frac{-\rho u}{\dot{m}_A}(1-\beta)}_{a}x\right)\right]$$

$$\frac{d\chi_A}{dx} = -\frac{j}{\dot{m}_A}$$

$$\chi_A = \int_0^x \frac{\rho u\chi_{A0}}{\dot{m}_A}\left[A-B\exp(-ax)\right]dx$$

$$= \frac{\rho u\chi_{A0}}{\dot{m}_A}\left[Ax-\frac{B}{-a}\left(\exp(-ax)-1\right)\right]$$

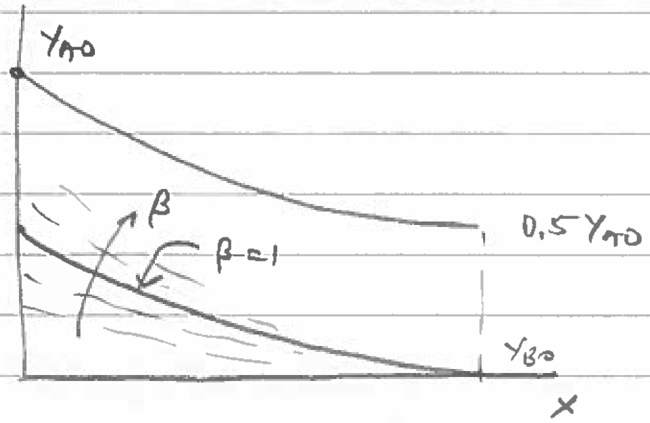
$$= \frac{\rho u\chi_{A0}}{\dot{m}_A}\left[(1-0.5\beta)x+\frac{0.5(1-\beta)}{\frac{u(1-\beta)}{Q_A}}\exp(-ax)\right]$$

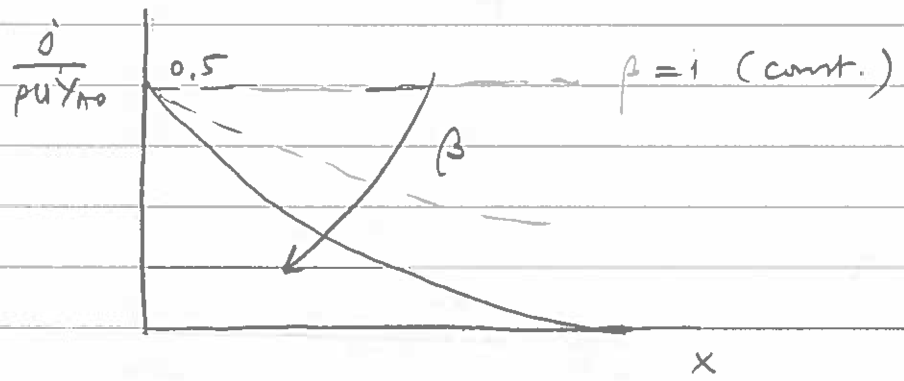
$$\chi_A = \frac{uL\chi_{A0}}{\dot{m}_A}\left[(1-0.5\beta)\frac{x}{L}-\frac{0.5Q_A}{uL}\exp(-ax)\right]$$

$$\chi_A(0) = \frac{uL\chi_{A0}}{\dot{m}_A}\left[0+\frac{0.5Q_A}{uL}(1)\right] = 0.5\chi_{A0}$$ ✓

②/4

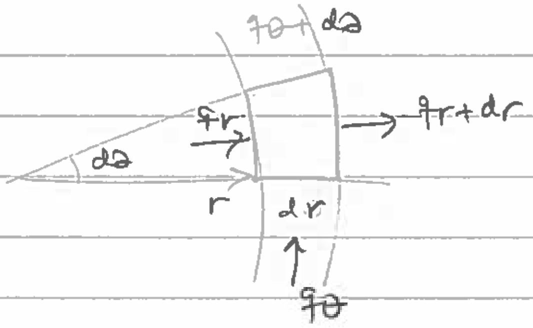
(e)


Y_{A0}
β
$\beta = 1$
$0.5\,Y_{A0}$
Y_{B0}
x


$\frac{\dot{O}}{\rho u Y_{A0}}$
0.5
$\beta = 1$ (const.)
β
x

Q3 CONDUCTION

(a) ENERGY BALANCE IN CYL. COORDS



$$d\theta\left(r q_r - (r+dr)\, q_{r+dr}\right) + \left(q_\theta - q_{\theta+d\theta}\right) dr + \dot{q}\, r\, d\theta\, dr$$

$$= \frac{\partial}{\partial t}\left[\rho c T\right] r\, d\theta\, dr$$

$\div\; r\, d\theta\, dr$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r q_r\right) + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

$$\left.\begin{aligned} q_r &= -\lambda \frac{\partial T}{\partial r} \\ q_\theta &= -\frac{\lambda}{r}\frac{\partial T}{\partial \theta} \end{aligned}\right\} \quad \frac{\lambda}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial \theta^2}\right] + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

$$\frac{\lambda}{\rho c}\,\underbrace{\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{r\,\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial \theta^2}\right]}_{\nabla^2 T} + \frac{\dot{q}}{\rho c} = \frac{\partial T}{\partial t}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c}}$$

q.e.d.

(b) $T(r,t) = f(t)\, e^{-\beta(t) r^2}$ 3/(2)

$$\frac{\partial T}{\partial r} = f(-2r)\beta\, e^{-\beta r^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[f(-2r^2)\beta e^{-\beta r^2}\right]$$

$$= \frac{f\beta(-2)}{r}\left[2r + r^2(-2r\beta)\right]e^{-\beta r^2}$$

$$= -4f\beta\left[1-\beta r^2\right]e^{-\beta r^2}$$

$$\frac{\partial T}{\partial t} = \left[f' - r^2\beta' f\right]e^{-\beta r^2}$$

$$\left[f' - r^2\beta' f\right]e^{-\beta r^2} = 4\alpha f\beta\left[\beta r^2 - 1\right]e^{-\beta r^2}$$

$$f' - r^2\beta' f - 4\alpha f\beta^2 r^2 + 4\alpha f\beta = 0$$

$$\left(f' + 4\alpha f\beta\right) - fr^2\left[\beta' + 4\alpha\beta^2\right] = 0$$

for all r:

$$\begin{cases} \beta' + 4\alpha\beta^2 = 0 \rightarrow \dfrac{\beta'}{\beta^2} = -4\alpha \rightarrow \boxed{\beta = \dfrac{1}{4\alpha t}} \\ f' + 4\alpha f\beta = 0 \rightarrow \dfrac{f'}{f} = -4\alpha\beta \end{cases}$$

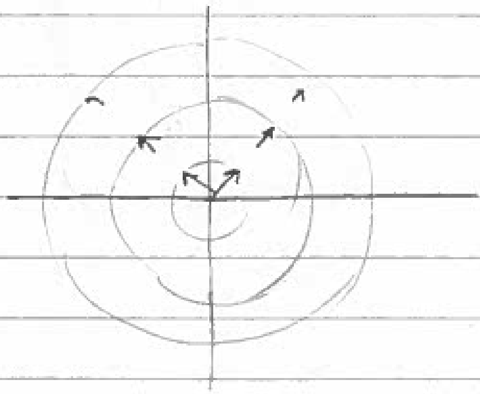
SUBSTITUTE β: $f' + 4\alpha\left(\dfrac{1}{4\alpha t}\right)f = 0$

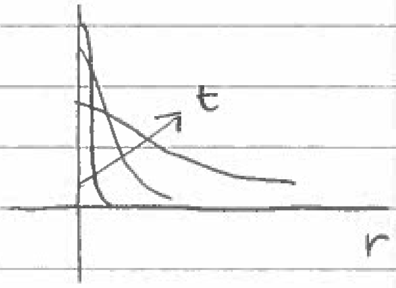
$$\frac{f'}{f} = -\frac{1}{t} \qquad \boxed{f = \frac{c}{t}}$$

3/3

(c) $T(r,t) = \frac{A}{\alpha t} \exp\left(-\frac{r^2}{4\alpha t}\right)$

(i) THIS IS A CYLINDRICALLY SYMMETRIC FIELD ABOUT $r=0$, WHICH SPREADS WITH TIME. THE HEAT FLUX LINES ARE RADIAL, AND THE KERNEL EXPANDS AND BECOMES SHALLOWER WITH TIME.





(ii) SINCE ENERGY IS CONSERVED, WE EXPECT THAT:

$$E = \int_0^\infty \rho c \frac{A}{\alpha t} \exp\left(-\frac{r^2}{4\alpha t}\right) 2\pi r \, dr = \text{const}$$

WE OBSERVE THAT $\frac{d}{dr}\left(\exp\left(-\frac{r^2}{4\alpha t}\right)\right) = \frac{-2r}{4\alpha t} \exp\left(-\frac{r^2}{4\alpha t}\right)$

SO THAT

$$E = \int_0^\infty -\rho c A \frac{4\alpha t}{\alpha t} \pi \, d\left(\exp(-\xi)\right) = 4\pi \rho c A$$

$$\boxed{A = \frac{E}{4\pi \rho c}}$$ ENERGY / LENGTH / heat capacity.

Q4 / RADIATION

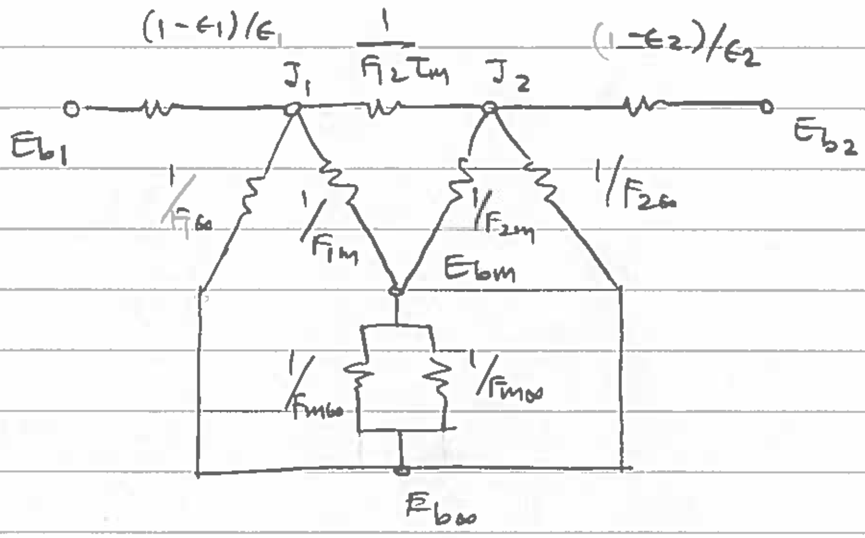
④/1

(a) 1 → 2 : $J_1 A F_{12} \tau_m$ RAD FROM 1→2 THROUGH m.

2 → 1 : $J_2 A F_{21} \tau_m$ RAD FROM 2→1 THROUGH m

$$q_{12} = (J_1 - J_2) F_{12} \tau_m = \frac{J_1 - J_2}{1/F_{12}\tau_m}$$

(b)



SYMMETRY:

$F_{1m} = F_{2m}$

$F_{1\infty} = F_{2\infty}$

$F_{12} = F_{21} = 0.3$

$F_{1m} + F_{1\infty} = 1$

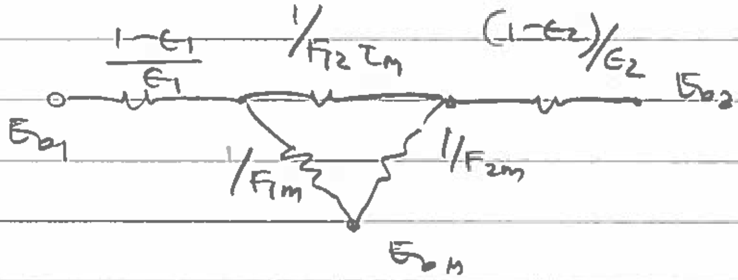
$0.4 + F_{1\infty} = 1 \quad \rightarrow F_{1\infty} = F_{2\infty} = 0.6$

$F_{m1} + F_{m\infty} = 1 \quad \rightarrow F_{m\infty} = 1 - F_{m1} = 0.7$

$\epsilon_m = 1 - \tau_m = 0.4$

(c) THE CIRCUIT IS SIMPLIFIED TO:

(i)



$\epsilon_1 = 0.8$; $\epsilon_2 = 0.7$ $\qquad F_{1m} = F_{2m} = 0.3$

$F_{12} = 0{,}4$

$$R_m = \frac{1}{F_{1m}} + \frac{1}{F_{m}} = \frac{2}{F_{1m}} = \frac{2}{0.3}$$

$$\frac{1}{R'} = \frac{1}{R_M} + \frac{1}{1/F_{12}\tau_m} = \frac{F_{1m}}{2} + \tau_m F_{12} = \frac{0.3}{2} + (0{,}6)(0{,}4) = 2{,}56$$

$$R_T = \frac{1-0{,}8}{0{,}8} + 2{,}56 + \frac{1-0{,}7}{0{,}7} = 3{,}24$$

$$q_{12} = \frac{E_{b1} - E_{b2}}{3{,}24}$$

(c) (ii) IN THIS CASE, WE CANNOT USE E_{b1} AND E_{b2} DIRECTLY, BUT INSTEAD INTEGRATE THE FRACTION OF THE ENERGIES LIMITED WITHIN THE CORRESPONDING WAVELENGTH

$$q_{12} = \int_0^{\lambda_0} \frac{E_{b1,\lambda} - E_{b2,\lambda}}{R_T(\lambda)} d\lambda +$$

$$\int_{\lambda_0}^{\infty} \frac{E_{b1,\lambda} - E_{b2,\lambda}}{R_T(\lambda)} d\lambda$$

THIS CAN BE DONE BY USING THE TABULATED FRACTION

$$F(\lambda, T) = \frac{\int_0^{\lambda} E_{b,\lambda} \, d\lambda}{E_{b}}$$

OR NUMERICALLY, $R_T(\lambda) = \frac{0.3}{2} + \varepsilon_m(0.4)$

$$\text{AND} \quad \varepsilon_m = \begin{cases} 0.5 & \lambda < \lambda_0 \\ 0.6 & \lambda > \lambda_0 \end{cases}$$