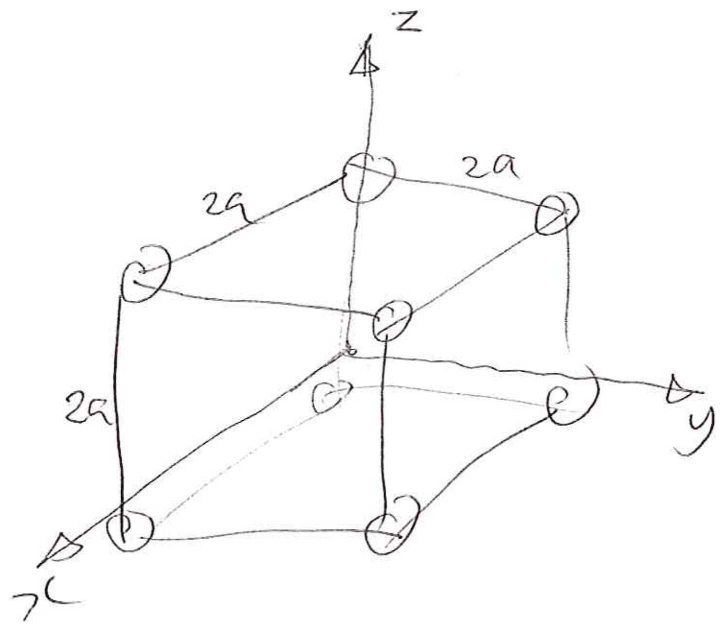
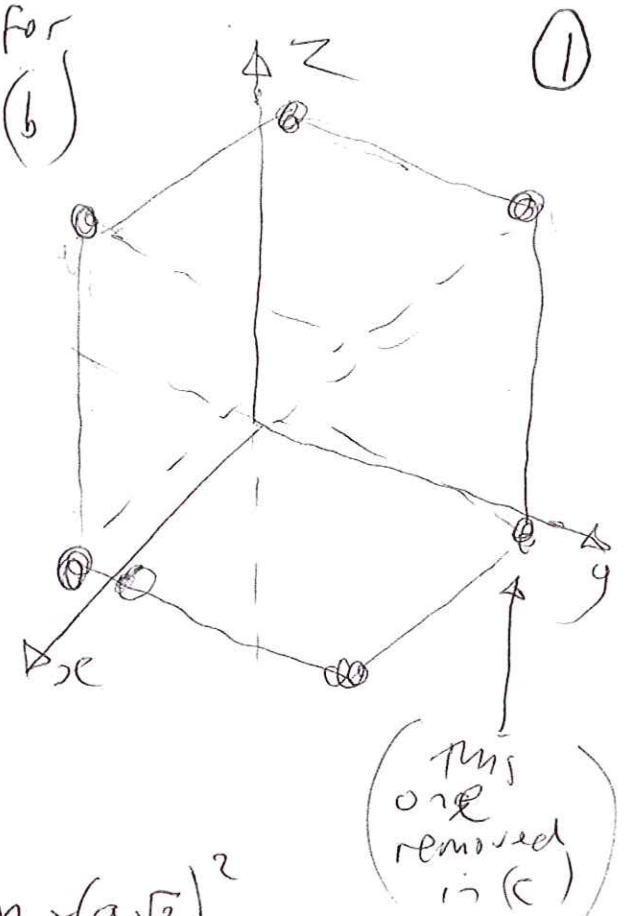
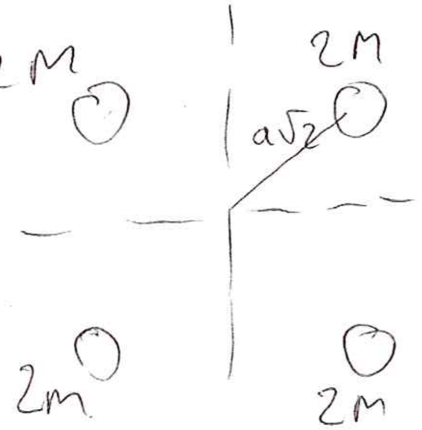
3C5 2015

1 (a)



for (b)





top view

$A = 4 \times 2M \times (a\sqrt{2})^2$

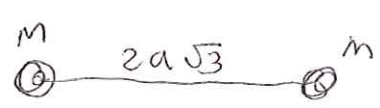
$= 16Ma^2$

AAA body $\therefore$ this is required result

(b) view along line $0 - (a,a,a)$, A is unchanged

& line $0 - (a,a,a)$ (call it C for AAC body)

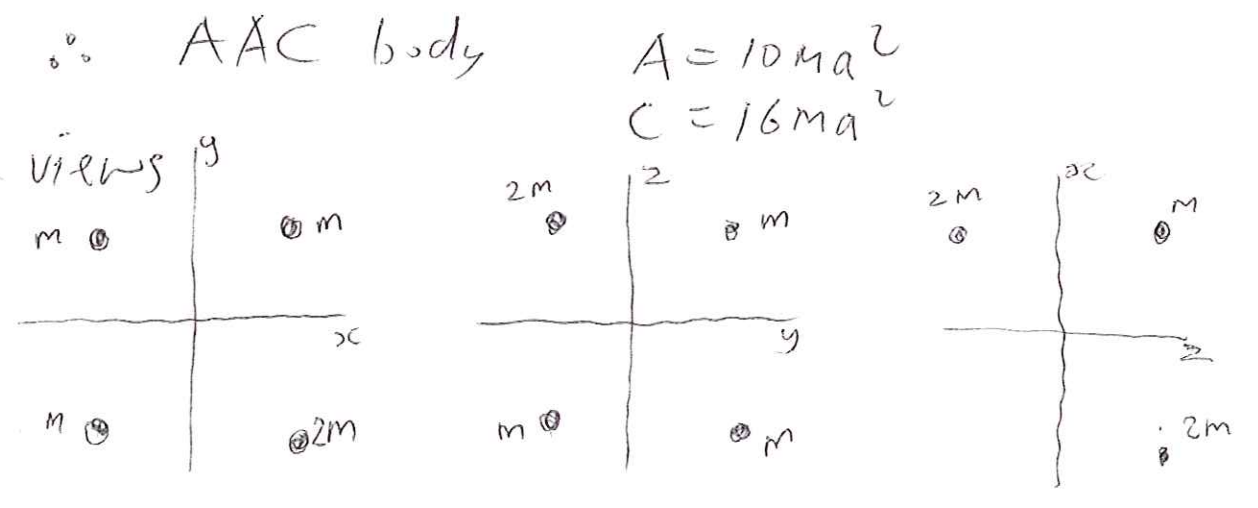
remove two masses



new $A = 16Ma^2 - 2m(a\sqrt{3})^2 = 10Ma^2$

$\therefore$ AAC body $A = 10Ma^2$
$C = 16Ma^2$

(c) three views



$$I_{xx} = I_{yy} = I_{zz} = 5m(a\sqrt{2})^2 = 10ma^2$$

$$I_{xy} = \sum m_i x_i y_i = -ma^2$$

$$I_{yz} = -ma^2$$

$$I_{xz} = -3ma^2$$

$\therefore$ Inertia matrix is

$$I = ma^2 \begin{bmatrix} 10 & 1 & 3 \\ 1 & 10 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

(d) Given eigen vector $= (1, 0, -1)$

then $I \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\therefore \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} ma^2 = \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\qquad \therefore \underline{\lambda = 7ma^2}$

or given $\lambda = 7ma^2$

then $I \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 7ma^2 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$

$\therefore$ $10 + a + 3b = 7$

$1 + 10a + b = 7a$ $\qquad \therefore b = -1 - 3a$

$\therefore 10 + a - 3 - 9a = 7$ $\qquad \therefore a = 0$

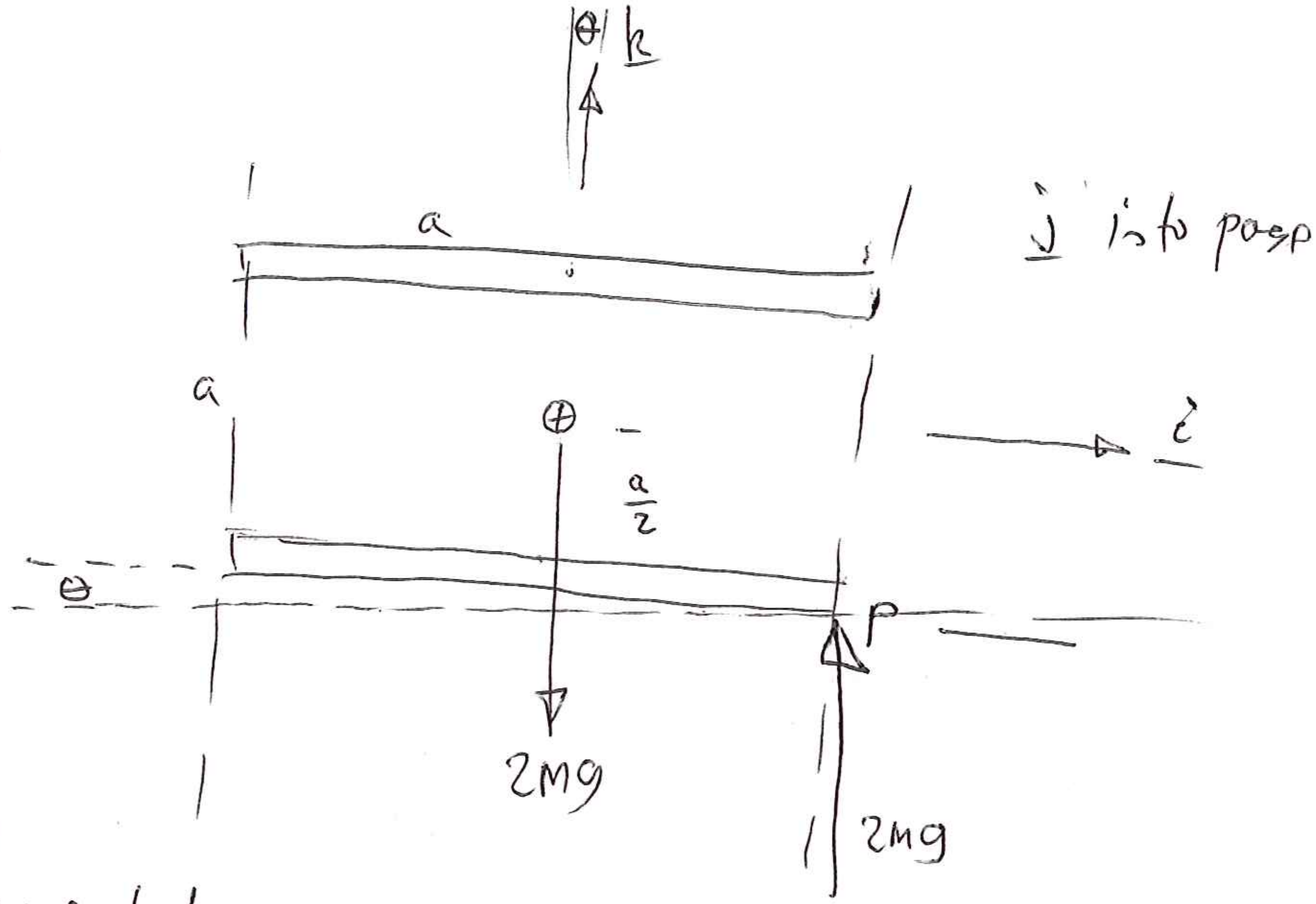
$\therefore b = 1$

$\therefore$ principal axis $= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Q1. Examiner's Comment:

Quite well done. Some candidates were not at all well prepared for this question, quite a few low answers. Several with 20/20 so the question was clearly do-able.

2



(a)

AAC body

$C = 2 \times \frac{1}{2} m a^2$

$= m a^2$

| a |

$A = 2\left[\frac{1}{4} m a^2 + m\left(\frac{a}{2}\right)^2\right] = m a^2$

(So it happens to be an AAA body, but that is not important)

(b)

Couple $Q_2 = -2mga$ about $\underline{j}$ axis for small θ

(c) Assume G is at rest.

No slip at P $\therefore$ $\underline{\omega} \times \left(-\frac{a}{2}\underline{k} + a\underline{i}\right) = 0$

$$\therefore \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ a & 0 & -\frac{a}{2} \end{vmatrix} = 0$$

$$\therefore \left.\begin{array}{l} -\frac{a}{2}\omega_2 = 0 \\ \frac{a}{2}\omega_1 + \omega_3 = 0 \\ -a\omega_2 = 0 \end{array}\right\} \quad \therefore \omega_2 = 0 \text{ and } \omega_3 = -\frac{a}{2}\omega_1$$

Euler angles $\therefore$ angular velocities

$$\Omega_1 = \omega_1 = -\dot\phi \sin\theta \simeq -\dot\phi\,\theta$$

$$\Omega_2 = \omega_2 = \dot\theta$$

$$\Omega_3 = \dot\phi\cos\theta \simeq \dot\phi$$

$$\omega_3 = -\frac{a}{2}\omega_1 \text{ (no slip)} = \frac{a}{2}\dot\phi\,\theta$$

Gyro 2 $\therefore$
steady state

$$\cancel{A\dot\Omega_2} + (A\Omega_3 - C\omega_3)\,\omega_1 = Q_2$$

$$\therefore \left(A\dot\phi - \cancel{C\frac{a}{2}\dot\phi\,\theta}\right)(-\dot\phi\,\theta) = -2mga$$

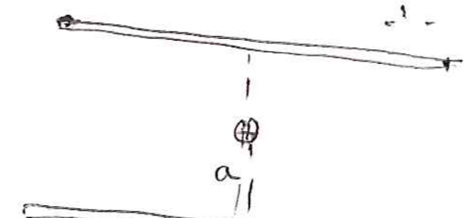
(neglect for small θ)

$$\therefore A\dot\phi^2\theta \simeq 2mga$$

$A = ma^2$ $\quad\therefore \dot\phi^2 \simeq \dfrac{2g}{a\theta}$

$$\therefore \dot\phi \simeq \sqrt{\frac{2g}{a\theta}}$$

(d)



Path circumference

$$C = 2\pi\left(a - \frac{a\theta}{2}\right)$$

Angle advance $\beta = \dfrac{2\pi a - C}{a}$

$$= \pi\theta$$

time for cycle $T = \dfrac{2\pi}{\dot\phi}$

$$\dot\beta = \frac{\beta}{T} = \frac{\beta\dot\phi}{2\pi} = \frac{\theta\dot\phi}{2} = \sqrt{\frac{g\theta}{2a}}$$

Q2. Examiner's Comment:

A printing irregularity was noticed during reading time. This was fixed and no candidates showed any sign of having been disadvantaged. Quite well done, but the geometry for part (d) escaped most candidates. It is all about making the right small-angle approximations.

3 (a) $\underline{Q} = \dot{\underline{h}}$

$$\underline{Q} = Q_1 \underline{i} + Q_2 \underline{j} + Q_3 \underline{k}$$
$$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$$
$$\underline{h} = A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k}$$
$$\dot{\underline{h}} = A\dot{\omega}_1 \underline{i} + B\dot{\omega}_2 \underline{j} + C\dot{\omega}_3 \underline{k}$$
$$+ \underbrace{(\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}) \times (A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k})}_{(\underline{\omega} \times \underline{h})}$$

$\underline{Q} = \dot{\underline{h}}$
$\underline{i}$ component

$$\boxed{\begin{aligned} Q_1 &= A\dot{\omega}_1 - (B-C)\omega_2\omega_3 \\ Q_2 &= B\dot{\omega}_2 - (C-A)\omega_3\omega_1 \\ Q_3 &= C\dot{\omega}_3 - (A-B)\omega_1\omega_2 \end{aligned}}$$

(b) Steady state $\underline{\omega} = \Omega \underline{j}$ (B- axis)
$\underline{Q} = 0$

$\omega_1 = 0 \qquad \omega_2 = \Omega \qquad \omega_3 = 0$

perturb: $\omega_1 = \omega_1'$ small
$\omega_2 = \Omega + \omega_2'$
$\omega_3 = \omega_3'$

$\dot{\omega}_1 = \dot{\omega}_1'$
$\dot{\omega}_2 = \dot{\omega}_2'$
$\dot{\omega}_3 = \dot{\omega}_3'$

$$\therefore \left.\begin{aligned} A\dot{\omega}_1' - (B-C)(\Omega + \omega_2')\omega_3' &= 0 \\ B\dot{\omega}_2' - (C-A)\omega_3'\omega_1' &= 0 \\ C\dot{\omega}_3' - (A-B)\omega_1'(\Omega + \omega_2') &= 0 \end{aligned}\right]$$

neglect terms of order $\omega_3'\omega_1'$

$$\therefore \left.\begin{aligned} A\dot{\omega}_1' - (B-C)\Omega\omega_3' &= 0 \\ C\dot{\omega}_3' - (A-B)\Omega\omega_1' &= 0 \end{aligned}\right]$$

$$\therefore \ddot{\omega}_1' + \frac{(B-C)(B-A)\Omega^2}{AC}\omega_1' = 0$$

or $\ddot{\omega}_1' + \lambda^2 \omega_1' = 0$

but since $\lambda^2 < 0$
the motion is unstable
if $\lambda^2 > 0$ then d.e. gives SHM

3(b)

(i) $$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = Q$$

$$\Rightarrow \frac{d}{dt}\left[2A(q)\dot{q}\right] - \dot{q}^2\frac{\partial A}{\partial q} + \frac{\partial V}{\partial q} = Q$$

$$\Rightarrow 2\frac{\partial A}{\partial q}\dot{q}^2 + 2A(q)\ddot{q} - \dot{q}^2\frac{\partial A}{\partial q} + \frac{\partial V}{\partial q} = Q$$

$$\Rightarrow \underline{2A(q)\ddot{q} + \frac{\partial A}{\partial q}\dot{q}^2 + \frac{\partial V}{\partial q} = Q} \qquad [15\%]$$

(ii) Power $P = Q\dot{q}$

$$\Rightarrow \underline{P = 2A(q)\ddot{q}\dot{q} + \frac{\partial A}{\partial q}\dot{q}^3 + \frac{\partial V}{\partial q}\dot{q}} \qquad [10\%]$$

(iii) For $Q=0$ then $P=0$

$$T = A(q)\dot{q}^2 \Rightarrow \frac{d}{dt}[T] = 2A(q)\dot{q}\ddot{q} + \frac{\partial A}{\partial q}\dot{q}^3$$

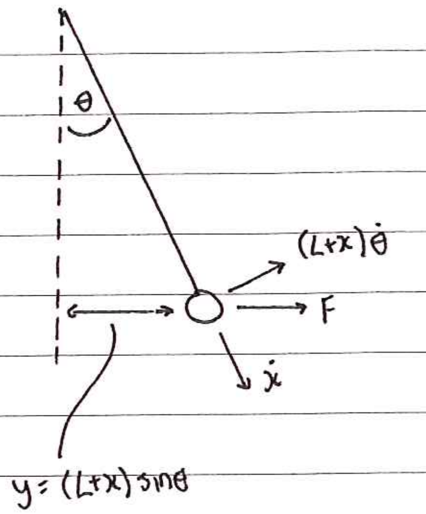
Also $\frac{d}{dt}[V] = \frac{\partial V}{\partial q}\dot{q}$

$$\Rightarrow P = 0 \Rightarrow \underline{\frac{d}{dt}[T+V] = 0} \qquad [25\%]$$

Q3. Examiner's Comment:
(a) Instability of a spinning rigid body. Bookwork, and many rolled out a correct answer. Disappointingly many didn't know where to start!.

(b) A good testing lagrange question on thebasics of Lagrange's equations. Many got lost in lazy algebra, most commonly not differentiating correctly using chain rule.

4 a)



$T = \frac{1}{2}M[(L+x)^2\dot{\theta}^2 + \dot{x}^2]$

$V = \frac{1}{2}kx^2 + Mg(L+x)(-\cos\theta)$

Spring + gravitational

$\delta W = F\delta y = F[\delta x \sin\theta + (x+L)\cos\theta . \delta\theta]$

$\Rightarrow \quad Q_x = F\sin\theta$

$Q_\theta = F(x+L)\cos\theta$

Lagrange for x: $\frac{\partial T}{\partial \dot{x}} = M\dot{x}$, $\frac{\partial T}{\partial x} = M(L+x)\dot{\theta}^2$, $\frac{\partial V}{\partial x} = kx - Mg\cos\theta$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_x$$

$$\Rightarrow \quad \underline{M\ddot{x} - M(L+x)\dot{\theta}^2 + kx = F\sin\theta + Mg\cos\theta}$$

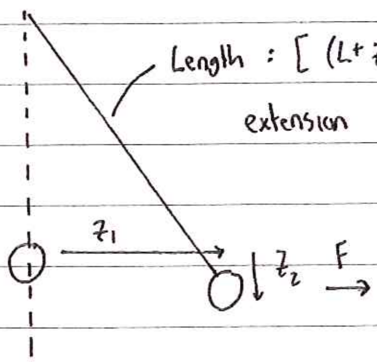
Lagrange for θ: $\frac{\partial T}{\partial \dot{\theta}} = M(L+x)^2\dot{\theta}$, $\frac{\partial T}{\partial \theta} = 0$, $\frac{\partial V}{\partial \theta} = Mg(L+x)\sin\theta$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

$$\Rightarrow \quad \underline{M(L+x)^2\ddot{\theta} + 2M(L+x)\dot{x}\dot{\theta} + Mg(L+x)\sin\theta = F(x+L)\cos\theta}$$

[30%]

b)



Length : $[(L+z_2)^2 + z_1^2]^{1/2}$

extension $\varepsilon = [(L+z_2^2) + z_1^2]^{1/2} - L$

$T = \frac{1}{2}M(\dot{z}_1^2 + \dot{z}_2^2)$

$V = -Mg(L+z_2) + \frac{1}{2}k\varepsilon^2$

$Q_2 = 0$

$Q_1 = F$

For z_1: $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{z}_1}\right] - \frac{\partial T}{\partial z_1} + \frac{\partial V}{\partial z_1} = Q_1$

$$\underline{M\ddot{z}_1 + k\varepsilon \underbrace{\frac{\partial \varepsilon}{\partial z_1}} = F}$$

$$\frac{z_1}{[(L+z_2)^2 + z_1^2]^{1/2}} \equiv \sin\theta \Rightarrow \text{This term is the resolved component of the tension}$$

For z_2: $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{z}_2}\right] - \frac{\partial T}{\partial z_2} + \frac{\partial V}{\partial z_2} = Q_2$

$$\underline{M\ddot{z}_2 + k\varepsilon \underbrace{\frac{\partial \varepsilon}{\partial z_2}} - Mg = 0} \qquad [30\%]$$

$$\frac{L+z_2}{[(L+z_2)^2 + z_1^2]^{1/2}} \equiv \cos\theta \Rightarrow \text{as above}$$

c) For conservation of generalised momentum: T must be independent of q

V must be independent of q

For $g = 0$ these conditions are met for $\theta \Rightarrow \frac{\partial T}{\partial \dot{\theta}} = \text{const}$

$$\Rightarrow \underline{M(L+x)^2\dot{\theta} = \text{const}} \qquad [20\%]$$

$\Rightarrow$ Conservation of moment of momentum about pivot.

d) For $\dot{\theta} = \Omega$ and $g = 0$ the x equation becomes:

$$M\ddot{x} - M(L+x)\Omega^2 + kx = 0$$

$$M\ddot{x} + (k - M\Omega^2)x = ML\Omega^2 \Rightarrow \omega_n^2 = \frac{k - M\Omega^2}{M} = \underline{\frac{k}{M} - \Omega^2}$$

For $\Omega^2 > k/M$ ω_n^2 is negative $\Rightarrow$ ω_n is imaginary $\Rightarrow$ unstable system. [20%]

Q4. Examiner's Comment:

A good Lagrange question. Many errors writing down potential energy, though KE seemed to be OK. Many did well getting to the answer in 4(b) but many explained that the pendulum rod would buckle – not sure how if it's under tension.