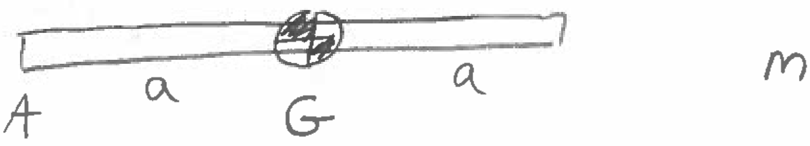
Part IIA Engineering Tripos

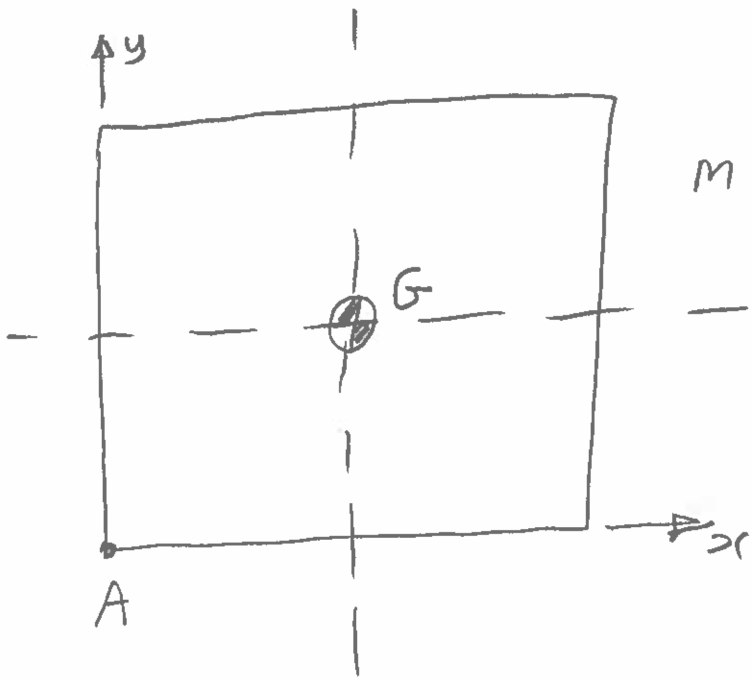
3C5 Dynamics 2016

1/(a)

Use



$I_G = \frac{1}{3} m a^2$ $\qquad I_A = \frac{4}{3} m a^2$



$I_G = \frac{2}{3} m a^2$ by perpendicular axis theorem

$I_A = \frac{8}{3} m a^2$ by parallel axis theorem

$$I_{xy} = \int_0^{2a}\int_0^{2a} xy \, dm$$

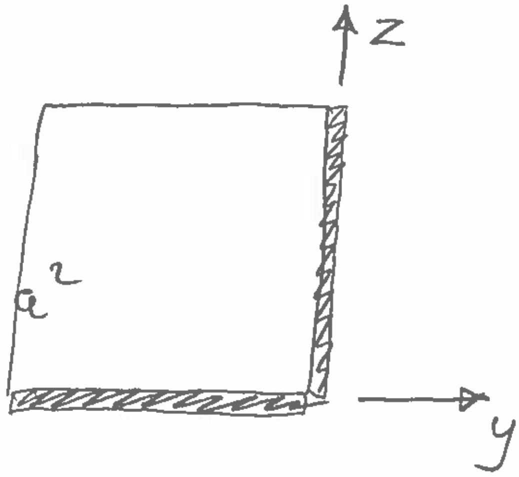
$$= \int_0^{2a} x \, dx \int_0^{2a} y \, dy \frac{m}{4a^2}$$

$$= \frac{(2a)^2}{2} \frac{(2a)^2}{2} \frac{m}{4a^2} = m a^2$$

Now assemble according to the various views, noting sign of "I_{xy}"

1(a) cont

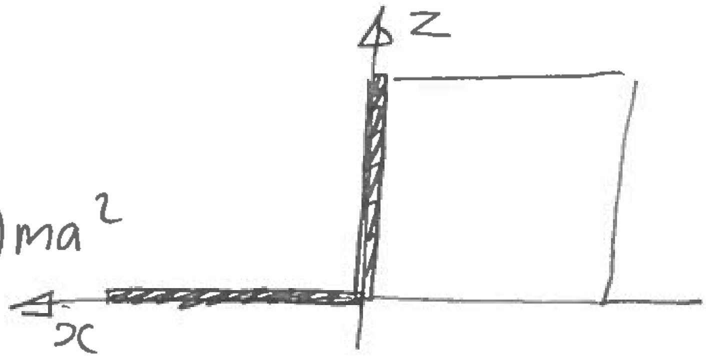
View along x



$$I_{xx} = \left(\frac{4}{3} + \frac{4}{3} + \frac{8}{3}\right) m a^2 = \frac{16}{3} m a^2$$

$$I_{yz} = \int_{-2a}^{0} y\,dy \int_{0}^{2a} z\,dz \,\frac{m}{4a^2} = -ma^2$$

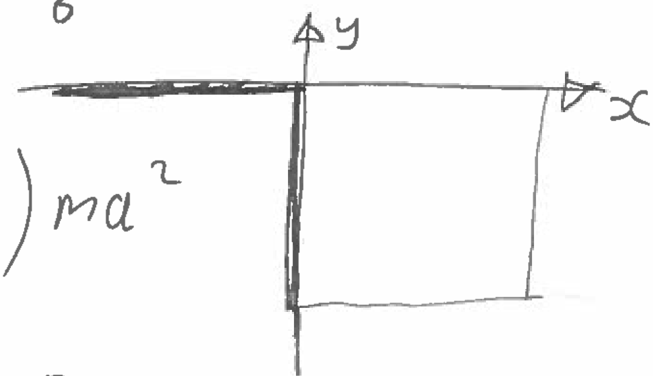
View along y



$$I_{yy} = \left(\frac{4}{3} + \frac{4}{3} + \frac{8}{3}\right) m a^2 = \frac{16}{3} m a^2$$

$$I_{xz} = \int_{-2a}^{0} x\,dx \int_{0}^{2a} z\,dz \,\frac{m}{4a^2} = -ma^2$$

View along z



$$I_{zz} = \left(\frac{4}{3} + \frac{4}{3} + \frac{8}{3}\right) m a^2 = \frac{16}{3} m a^2$$

$$I_{xy} = \int_{0}^{2a} x\,dx \int_{-2a}^{0} y\,dy \,\frac{m}{4a^2} = -ma^2$$

Put these into inertia matrix form:

1(a) cont

$$I = \frac{ma^2}{3}\begin{bmatrix} 16 & +3 & +3 \\ +3 & 16 & +3 \\ +3 & +3 & 16 \end{bmatrix}$$

(b) try $\underline{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $I\underline{u}_1 = \frac{ma^2}{3}\begin{bmatrix} 13 \\ -13 \\ 0 \end{bmatrix}$

$$= \frac{13ma^2}{3}\underline{u}_1$$

$$\therefore \lambda_1 = \frac{13ma^2}{3}$$

try $\underline{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $I\underline{u}_2 = \frac{ma^2}{3}\begin{bmatrix} 13 \\ 13 \\ -26 \end{bmatrix}$

$$= \frac{13ma^2}{3}\underline{u}_2$$

$$\therefore \lambda_2 = \frac{13ma^2}{3}$$

Q1 (cont) Orthogonality $\underline{u}_3 = \underline{u}_1 \times \underline{u}_2$ (4)

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

or $\underline{u}_3 = (1, 1, 1)$

$$[I]\,\underline{u}_3 = \frac{ma^2}{3}\begin{bmatrix} 22 \\ 22 \\ 22 \end{bmatrix} = \frac{22ma^2}{3}\underline{u}_3 \quad \therefore \lambda_3 = \frac{22}{3}ma^2$$

(c) G is at $\left[0 \;\; \frac{-2a}{3} \;\; \frac{2a}{3}\right]$ and note that $[0 \;\; -1 \;\; 1]$ is principal at O

(Check : $[I]\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \frac{ma^2}{3}\begin{bmatrix} 0 \\ -13 \\ 13 \end{bmatrix} = \frac{13ma^2}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

which is consistent with the body being "AAC"]

(Another check is that $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is $\perp$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$)

So it's easy to use parallel axis theorem twice along the $[0 \;\; -1 \;\; 1]$ axis to get to P at $[0 \;\; -a \;\; a]$

First $O \to G$ using $\overrightarrow{GO} = \left[0 \;\; \frac{2a}{3} \;\; \frac{-2a}{3}\right]$

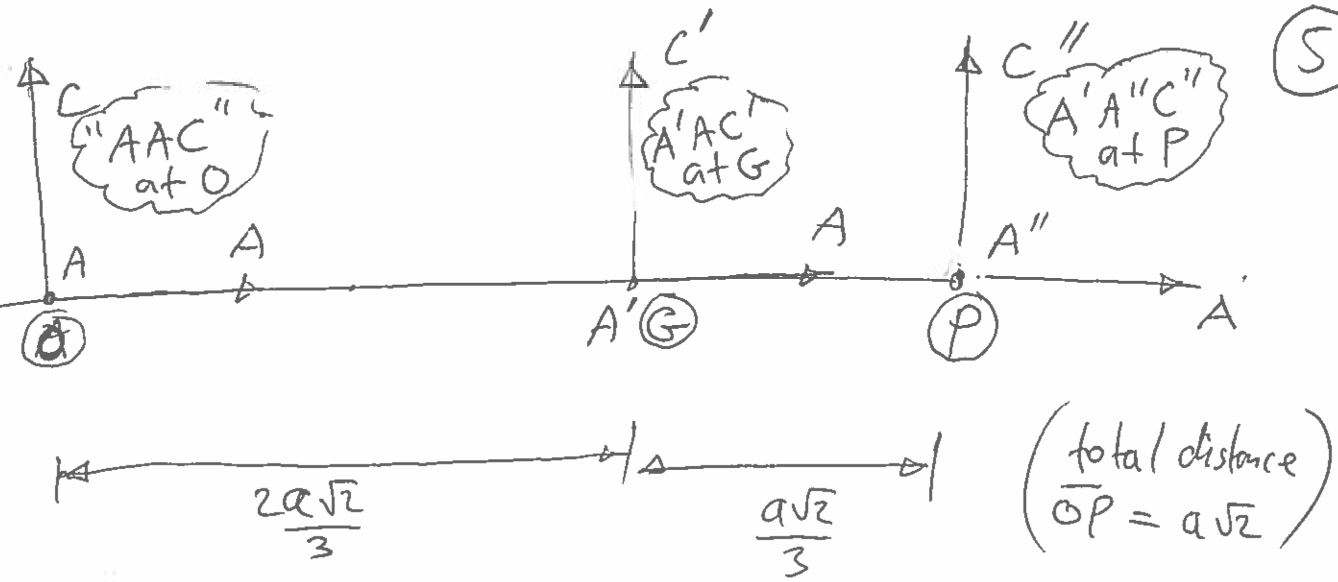
then $G \to P$ using $\overrightarrow{GP} = \left[0 \;\; \frac{-a}{3} \;\; \frac{a}{3}\right]$

but do this along $[0 \;\; -1 \;\; 1]$

$O \to G$ distance $-\frac{2\sqrt{2}a}{3}$

$G \to P$ distance $\frac{a\sqrt{2}}{3}$

Q1(c) cont



One principal axis $A = \frac{13}{3} ma^2$ is unchanged

C and A first go to C' and A' by subtracting $3m\left(\frac{2a\sqrt{2}}{3}\right)^2$ then to C'' and A'' by adding $3m\left(\frac{a\sqrt{2}}{3}\right)^2$

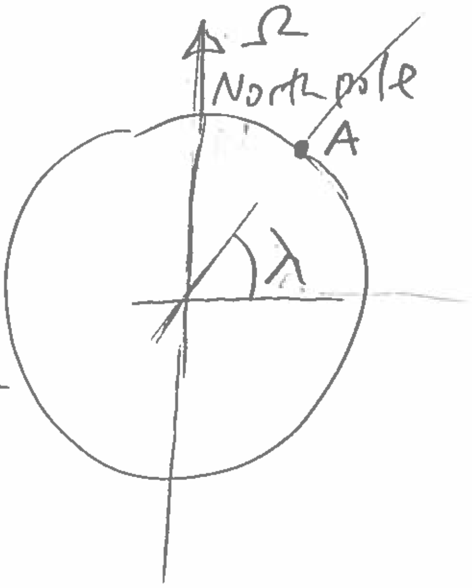
So
$$C'' = C + 3m\left(-\left(\frac{2a\sqrt{2}}{3}\right)^2 + \left(\frac{a\sqrt{2}}{3}\right)^2\right)$$
$$= C - 2ma^2$$
$$= \left(\frac{22}{3} - 2\right) ma^2 = \frac{16}{3} ma^2$$

and
$$A'' = A - 2ma^2$$
$$= \left(\frac{13}{3} - 2\right) ma^2 = \frac{7}{3} ma^2$$

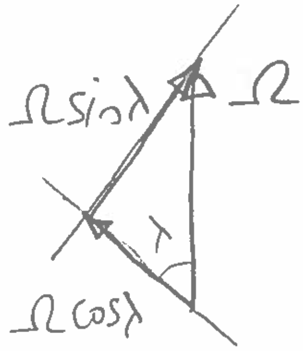
So principal moments of inertia at P

are
$$\left(\underset{A''}{\frac{7}{3}},\ \underset{A}{\frac{13}{3}} \text{ and } \underset{C''}{\frac{16}{3}}\right) ma^2$$

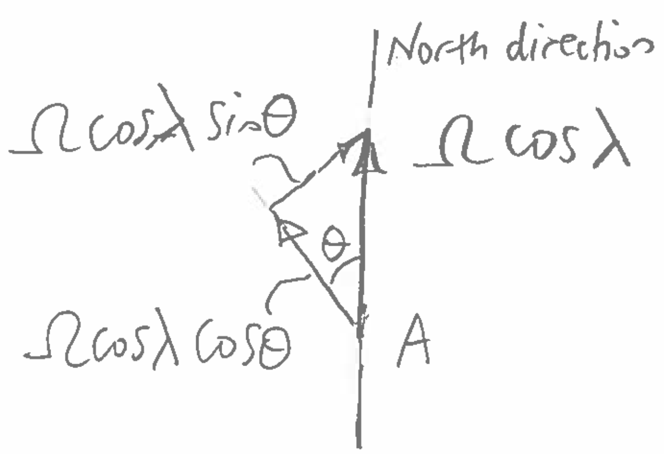
2/ (a)

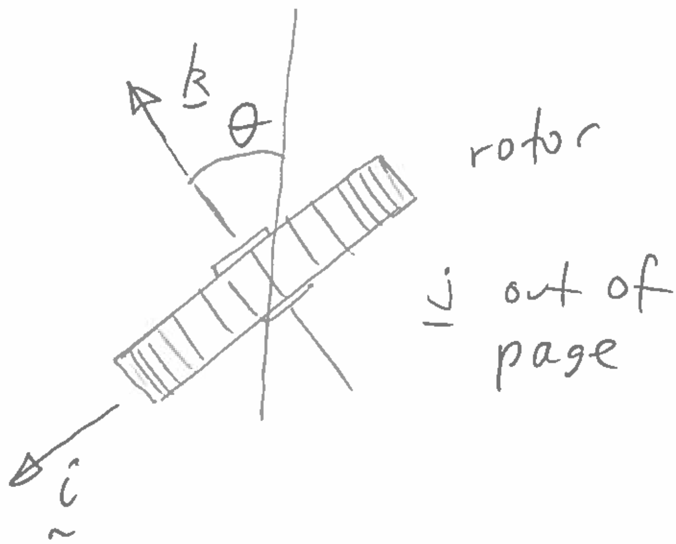


side view of Earth



Top view at A (observing point)





Angular velocity of Earth in $\underline{i}\ \underline{j}\ \underline{k}$ frame is

$$-\Omega \cos\lambda \sin\theta\, \underline{i} + \Omega \sin\lambda\, \underline{j} + \Omega \cos\lambda \cos\theta\, \underline{k}$$

Hence
$$\Omega_1 = -\Omega \cos\lambda \sin\theta$$
$$\Omega_2 = \Omega \sin\lambda + \dot{\theta}$$
$$\Omega_3 = \Omega \cos\lambda \cos\theta$$

2(b) Use Gyroscope equations noting that gimbal is free to turn about $\underline{j}$ axis hence $Q_2 = 0$

Gyro equation "2"

$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$$

Note: Earth angular velocity is constant

$\therefore \dot{\Omega}_2 = \ddot{\theta}$ and for fast spin

$C\omega_3 \gg A\Omega_3$ and $\omega_3 \simeq \omega$

$$\therefore \quad A\ddot{\theta} + C\omega\Omega\cos\lambda\sin\theta \simeq 0$$

(c) steady state solution when $\theta = \text{const}$

$$\therefore \sin\theta = 0$$

$$\therefore \theta = 0 \text{ or } \pi$$

Try $\theta \simeq 0$ $\quad \therefore \ddot{\theta} + \dfrac{C\omega\Omega\cos\lambda}{A}\theta \simeq 0$

This is SHM hence stable (use this result in (d))

Try $\theta = \pi + x$ for small x

$$\therefore \sin\theta = -\sin x \simeq -x$$

and $\ddot{\theta} = \ddot{x}$

2(c) cont So

$$\ddot{x} - \frac{C\omega\Omega\cos\lambda}{A}x = 0$$

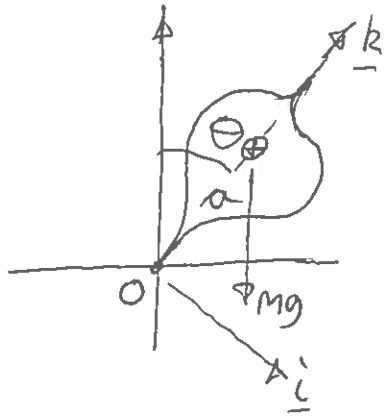
This leads to $e^{\alpha t}$ solutions hence unstable.

The stable solution around $\theta = 0$ means that the gyrocompass oscillates around true north. The device needs some damping to make it settle down to $\theta = 0$

(d) from (c) take SHM solution

$\therefore$ natural frequency (in rad/s) is $\sqrt{\frac{C\omega\Omega\cos\lambda}{A}}$ for small oscillations about true north

Q3(a)



$\omega_1 = \Omega_1 = -\dot{\phi}\sin\theta$

$\omega_2 = \Omega_2 = \dot{\theta}$

$\Omega_3 = \dot{\phi}\cos\theta$

$\omega_3 = \Omega_3 + \dot{\psi}$

= absolute angular velocity

$\underline{Q} = mga\sin\theta\,\underline{j}$

Use Gyro equation (2)

$$\text{"}A\dot{\omega}_2 + (A\Omega_3 - C\omega_3)\omega_1 = Q_2\text{"} \; (= mga\sin\theta) \quad (1)$$

But note A, A, C given at G so need to use parallel axis theorem $(A+ma^2), (A+ma^2), C$ for moments of inertia at O

(i) steady state fast spin (1) becomes

$$C\omega_3\dot{\phi}\sin\theta = mga\sin\theta \quad \therefore \boxed{\dot{\phi} = \frac{mga}{C\omega_3}}$$

$$T = \frac{2\pi}{\dot{\phi}}$$

(ii) steady state <u>not</u> fast spin (1) becomes

$$\left((A+ma^2)\dot{\phi}\cos\theta - C\omega_3\right)\dot{\phi}\sin\theta + mga\sin\theta = 0$$

small θ $\therefore$ $(A+ma^2)\dot{\phi}^2 - C\omega_3\dot{\phi} + mga = 0$

solution only if "$b^2 - 4ac$" ≥ 0 $\therefore (C\omega_3)^2 \geq 4(A+ma^2)mga$

$$\therefore \omega_3 \geq \sqrt{\frac{4(A+ma^2)mga}{C^2}} \quad \text{for stable spin}$$

3(b) (i) Definition of generalised momentum

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left[\frac{1}{2}\sum m_j \dot{q}_j^2\right]$$

$$= m_j \dot{q}_j$$ as expected for a particle

(ii) Lagrange $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$

$$\therefore \; m_j \ddot{q}_j + \frac{\partial V}{\partial q_j} = 0$$

(iii) $H = T + V$

$$= \frac{1}{2}\sum \frac{p_j^2}{m_j} + V(q_1, q_2 \ldots q_j \ldots)$$

So $\frac{\partial H}{\partial p_j} = \frac{p_j}{m_j} = \dot{q}_j$ <u>shown</u>

and $\frac{\partial H}{\partial q_j} = \frac{\partial V}{\partial q_j} = -m_j \ddot{q}_j$ (from (ii))

$$= -\frac{d}{dt}(m_j \dot{q}_j)$$

$$= -\dot{p}_j$$ <u>shown</u>

3b(iv) $H = T + V = \text{const}$

need to show $\dfrac{dH}{dt} = 0$

$$\frac{d}{dt}(H) = \sum_j \left[\frac{\partial H}{\partial p_j} \dot{p}_j + \frac{\partial H}{\partial q_j} \dot{q}_j \right]$$

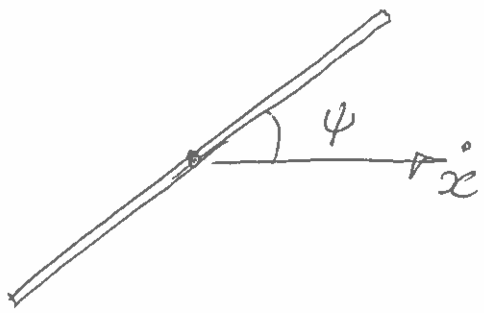
$$= \sum_j \left[\dot{q}_j \dot{p}_j - \dot{p}_j \dot{q}_j \right]$$

$$= 0$$

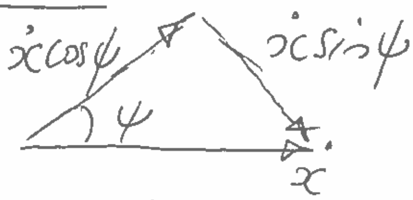
so H is const

$\therefore T + V = \text{const}$

4/(a)

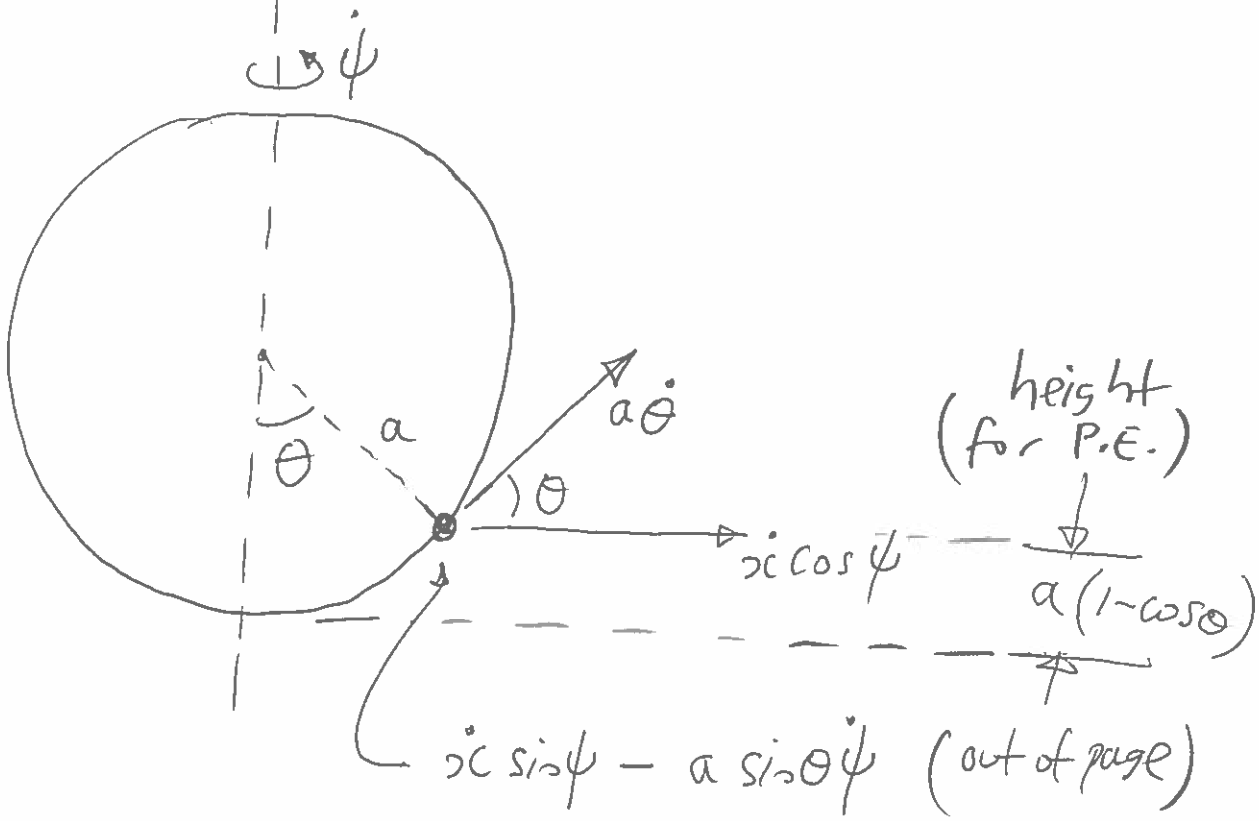


Top view

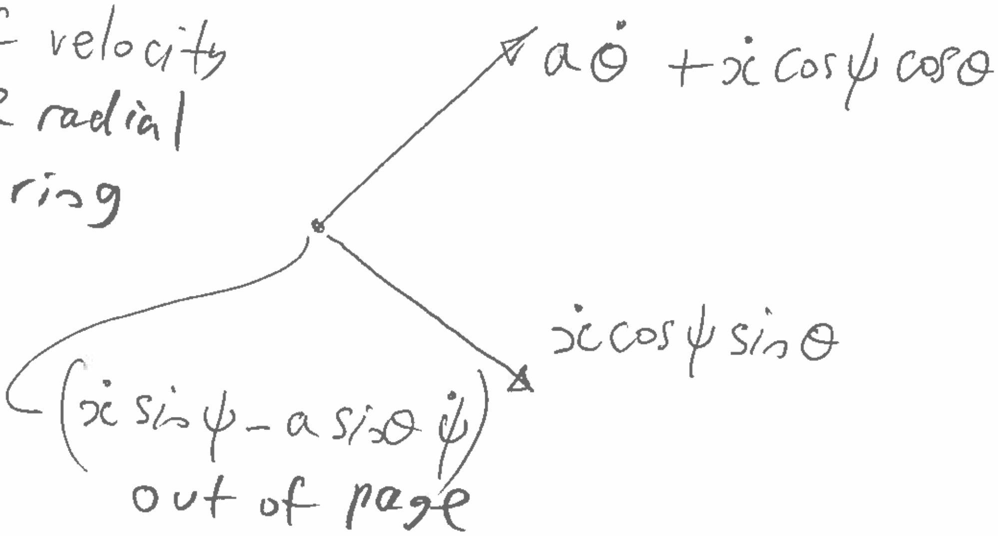


resolve $\dot{x}$ into plane of ring

plane of ring



Components of velocity tangential & radial and $\perp$ to ring



hence

4(a) cont

$$T_{bead} = \tfrac{1}{2} m\Big((a\dot\theta + \dot x \cos\psi\cos\theta)^2 + (\dot x\cos\psi\sin\theta)^2 + (\dot x \sin\psi - a\sin\theta\,\dot\psi)^2 \Big)$$

$$= \tfrac{1}{2} m\Big(a^2\dot\theta^2 + \dot x^2\cos^2\psi + 2a\dot x\dot\theta\cos\psi\cos\theta + (a\sin\theta\,\dot\psi - \dot x\sin\psi)^2 \Big)$$

as requested

Total $T = \tfrac{1}{2} M \dot x^2 + T_{bead}$

(b) Generalized momentum: T depends on θ & ψ but not on x so it is possible to define a generalized momentum in x

$$p = \frac{\partial T}{\partial \dot x} = (M+m)\dot x + ma\dot\theta\cos\psi\cos\theta - ma\dot\psi\sin\psi\sin\theta$$

(c) $$V = mga(1-\cos\theta) + \tfrac{1}{2}kx^2$$

$$\frac{\partial V}{\partial x} = kx \qquad \frac{\partial V}{\partial \theta} = mga\sin\theta$$

$$\frac{\partial T}{\partial \dot x} = (M+m)\dot x + ma\dot\theta\cos\psi\cos\theta - ma\dot\psi\sin\psi\sin\theta$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial \dot\theta} = ma^2\dot\theta + ma\dot x\cos\psi\cos\theta$$

$$\frac{\partial T}{\partial \theta} = -ma\dot x\dot\theta\cos\psi\sin\theta + ma\cos\theta\,\dot\psi\,(a\sin\theta\,\dot\psi - \dot x\sin\psi)$$

4c cont

Lagrange for x

$$\frac{d}{dt}\left((M+m)\dot{x} + ma\dot{\theta}\cos\psi\cos\theta - ma\dot{\psi}\sin\psi\sin\theta\right)$$
$$+ kx \quad = 0$$

and note that $\dot{\psi} = \text{const} \; (= \Omega)$

$$\therefore \boxed{(M+m)\ddot{x} + ma\ddot{\theta}\cos\psi\cos\theta - ma\dot{\theta}^2\cos\psi\sin\theta - ma\dot{\theta}\dot{\psi}\sin\psi\cos\theta + kx = 0} \quad \boxed{1}$$

Lagrange for θ

$$\frac{d}{dt}\left[ma^2\dot{\theta} + ma\dot{x}\cos\psi\cos\theta\right] \leftarrow \left[\begin{array}{l} ma^2\ddot{\theta} \\ + ma\ddot{x}\cos\psi\cos\theta \\ - ma\dot{x}\dot{\theta}\cos\psi\sin\theta \\ - ma\dot{x}\dot{\psi}\sin\psi\cos\theta \end{array}\right.$$

$$+ ma\dot{x}\dot{\theta}\cos\psi\sin\theta - ma^2\dot{\psi}^2\sin\theta\cos\theta$$
$$+ ma\dot{x}\dot{\psi}\sin\psi\cos\theta + mga\sin\theta = 0$$

$$\therefore \boxed{a\ddot{\theta} + \ddot{x}\cos\psi\cos\theta - a\dot{\psi}^2\sin\theta\cos\theta + g\sin\theta = 0} \quad \boxed{2}$$

and there is no equation for ψ other than $\dot{\psi} = \text{const} = \Omega$

For "static" solutions

$x = \text{const}$ $\qquad\qquad \theta = \text{const}$

$\therefore \dot{x} = \ddot{x} = 0 \qquad\qquad \dot{\theta} = \ddot{\theta} = 0$

$\therefore$ $\boxed{1}$ is satisfied

and $\boxed{2}$ gives $\quad -a\dot{\psi}^2 \sin\theta\cos\theta + g\sin\theta = 0$

$$\therefore \left(g - a\dot{\psi}^2\right)\sin\theta = 0$$

for which there are two solutions

$\theta = 0$ and $\theta = \pi$

For small oscillations the θ equation becomes

$$\theta \simeq 0 \qquad a\ddot{\theta} + \cos\psi\,\ddot{x} + \left(g - a\dot{\psi}^2\right)\theta = 0$$

$$\theta = \pi + z \qquad a\ddot{z} - \cos\psi\,\ddot{x} - \left(g + a\dot{\psi}^2\right)z = 0$$

Stiffness term

For small θ, stiffness is positive for $g - a\dot{\psi}^2 > 0$

for θ around π, stiffness is always negative

So the $\theta = 0$ solution is the only stable one and then only for $g - a\dot{\psi}^2 > 0$

(d) put $\boxed{1}$ & $\boxed{2}$ into matrix form

$$\begin{bmatrix} M+m & ma\cos\psi \\ \cos\psi & a \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -ma\dot{\psi}\sin\psi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & g-a\dot{\psi}^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = 0$$

with $\psi = \Omega t$ $\quad \dot{\psi} = \Omega$

Note time-varying mass matrix and damping matrix

HEM Hunt
May 2016