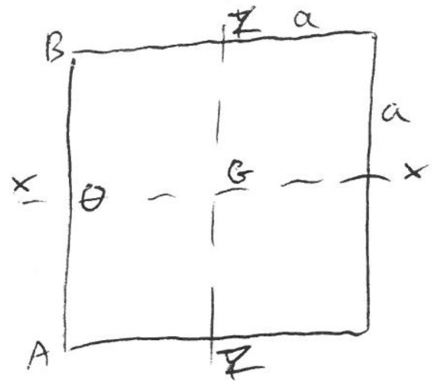
Module 3C5 2017 Solutions

①

1(a)



From Databook

$$I_{xx} = \frac{1}{3}ma^2 = I_{zz}$$

$$I_{yy} = I_{xx} + I_{zz} = \frac{2}{3}ma^2$$

Parallel axis theorem to O

$$\therefore I_O = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

with

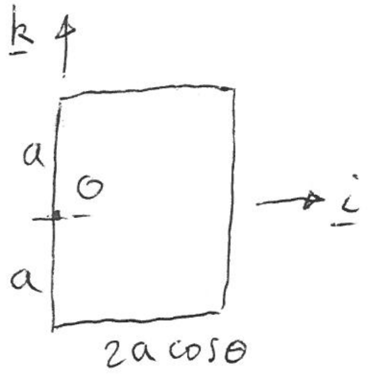
$$A = \frac{1}{3}ma^2$$

$$B = \frac{2}{3}ma^2 + ma^2 = \frac{5}{3}ma^2$$

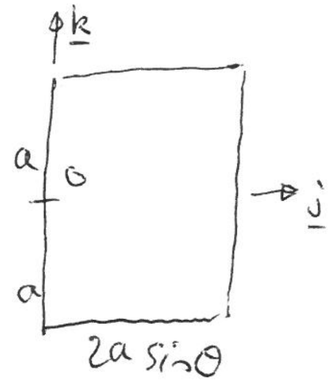
$$C = \frac{1}{3}ma^2 + ma^2 = \frac{4}{3}ma^2$$

(b)

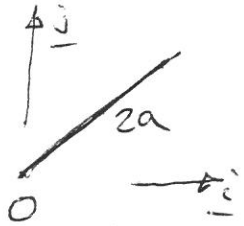
view $\underline{i}$ $\underline{k}$



view $\underline{j}$ $\underline{k}$



view $\underline{i}$ $\underline{j}$



$$I_{yy} = \frac{4}{3}m(a\cos\theta)^2 + \frac{1}{3}ma^2$$

$$= \frac{1}{3}ma^2(1 + 4\cos^2\theta)$$

$I_{xx} =$ similarly

$$= \frac{1}{3}ma^2(1 + 4\sin^2\theta)$$

$$I_{zz} = \frac{\cancel{4}}{3}ma^2$$

$$I_{xz} = 0 \qquad I_{yz} = 0$$

$$I_{xy} = \int xy \, dm$$

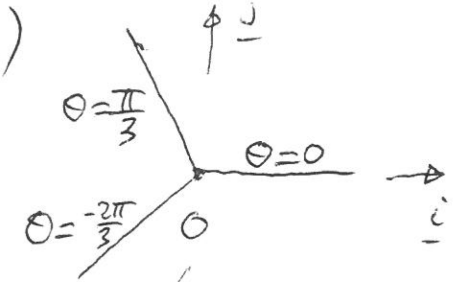
$$= \int_0^{2a} r\cos\theta \, r\sin\theta \frac{m}{2a} d$$

$$= \frac{m\cos\theta\sin\theta}{2a}\left[\frac{r^3}{3}\right]_0^{2a}$$

$$= \frac{4ma^2}{3}\cos\theta\sin\theta$$

$$\therefore I_O = \begin{bmatrix} 1+4\sin^2\theta & -4\cos\theta\sin\theta & 0 \\ -4\cos\theta\sin\theta & 1+4\cos^2\theta & 0 \\ 0 & 0 & 4 \end{bmatrix} \frac{ma^2}{3}$$

1. cont (c)



Expect AAC because of three axis symmetry
Three identical eigenvalues

$$I_0 = \frac{ma^2}{9}\left(\underset{\theta=0}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}} + \underset{\theta=\frac{2\pi}{3}}{\begin{bmatrix} 1+4\left(\frac{\sqrt{3}}{2}\right)^2 & -4\left(\frac{-1}{2}\right)\frac{\sqrt{3}}{2} & 0 \\ -4\left(\frac{-1}{2}\right)\frac{\sqrt{3}}{2} & 1+4\left(\frac{-1}{2}\right)^2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}\right.$$

$$\left. + \underset{\theta=\frac{-2\pi}{3}}{\begin{bmatrix} 1+4\left(-\frac{\sqrt{3}}{2}\right)^2 & -4\left(\frac{-1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) & 0 \\ -4\left(\frac{-1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) & 1+4\left(\frac{-1}{2}\right)^2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}\right)$$

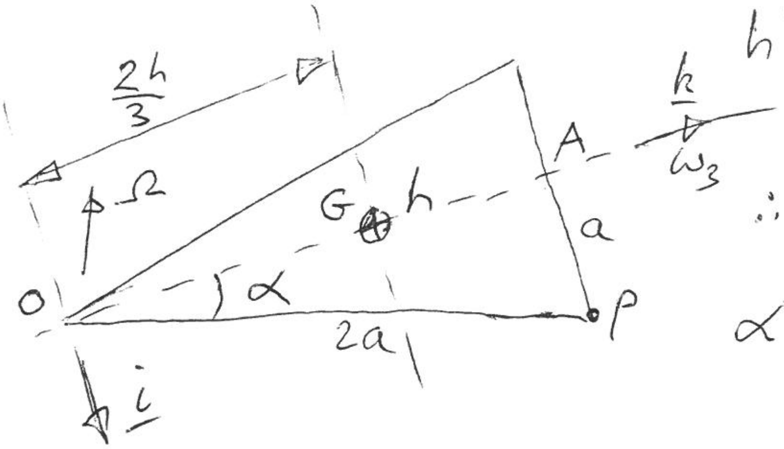
$$= \frac{ma^2}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & \cancel{4}12 \end{bmatrix} = ma^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

(d) For N plates of mass m is total

$$I_0 = ma^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$ because if the three plates is AAC and indistinguishable from a cylinder of mass m then this must be so for N plates equispaced.

Q2 Conical shell

a/



$h^2 + a^2 = (2a)^2$

$\therefore\ h = \sqrt{3}\, a$

$\alpha = 30°$

$OG = \frac{2}{3} h = \frac{2}{3}\sqrt{3}\, a$

AAC body

$$A = m\left(\frac{a^2}{4} + \frac{h^2}{18}\right) + m\,\overline{OG}^2$$

$$= ma^2\left(\frac{1}{4} + \frac{3}{18} + \frac{4}{9}3\right)$$

$$\boxed{C = \frac{ma^2}{2}}$$

$$= ma^2\left(\frac{3}{12} + \frac{2}{12} + \frac{16}{12}\right)$$

$$\boxed{A = \frac{7}{4} ma^2}$$

b/ No slip at P

$\underline{v}_P = \underline{v}_A + \omega_3 \underline{k} \times a\underline{i} = 0$

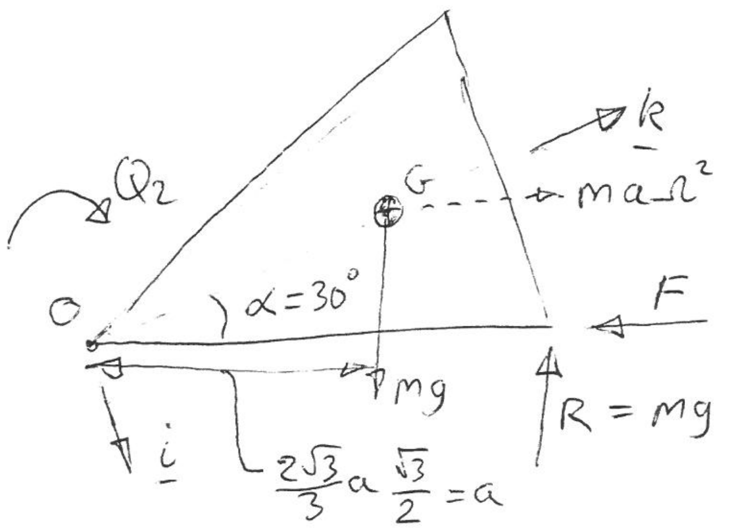
(note $\overline{OA}\cos\alpha$ is needed to get V_A)

$\Rightarrow \Omega h \cos\alpha + \omega_3 a = 0$

$\therefore\ \omega_3 = -\Omega\sqrt{3}\frac{\sqrt{3}}{2}$

$= -\frac{3\Omega}{2}$

c/



$Q_2 = -mg\,2a$

$+ mga$

$= -mga$

Gyroscope equations, Q_2 equation is useful

$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$

$$\Omega_1 = -\dot{\phi}\sin\theta$$
$$\Omega_2 = \dot{\theta}$$
$$\Omega_3 = \dot{\phi}\cos\theta$$

with $\dot{\phi} = \Omega$

$\theta = \frac{\pi}{2} - \alpha \qquad \dot{\theta} = 0$

$$\therefore \Omega_1 = -\Omega\cos\alpha = -\Omega\frac{\sqrt{3}}{2}$$
$$\Omega_2 = 0$$
$$\Omega_3 = \Omega\sin\alpha = \Omega/2$$

$$\therefore -mga = 0 + \left(\frac{7}{4}ma^2\frac{\Omega}{2} - \frac{ma^2}{2}\left(\frac{-3\Omega}{2}\right)\right)\left(-\Omega\frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{7}{2} + 3\right)ma^2\Omega^2\frac{\sqrt{3}}{8}$$

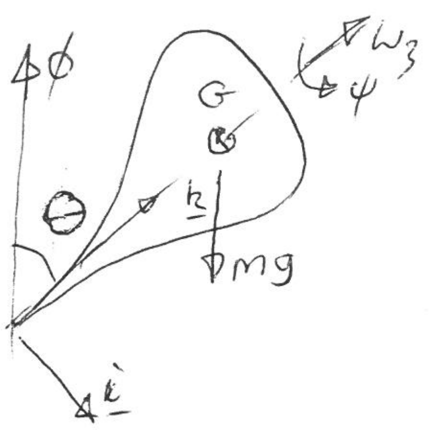
$$\therefore \Omega^2 = \frac{16}{13\sqrt{3}}\frac{g}{a} = \frac{16\sqrt{3}}{39}\frac{g}{a}$$

d/ $$\mu = \frac{F}{R} = \frac{ma\Omega^2}{mg}$$

$$= 16\sqrt{3}/39$$

Q3

a/



$$\omega_1 = \Omega_1 = -\dot\phi \sin\theta$$

$$\omega_2 = \Omega_2 = \dot\theta$$

$$\Omega_3 = \dot\phi\cos\theta$$

$$\omega_3 = \Omega_3 + \dot\psi$$

b/ $$T = \tfrac{1}{2}A(\dot\phi\sin\theta)^2 + \tfrac{1}{2}A\dot\theta^2 + \tfrac{1}{2}C(\dot\phi\cos\theta + \dot\psi)^2$$

$$V = mga(\cos\theta - 1)$$

Lagrange: $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

ψ: $$\frac{\partial T}{\partial \dot\psi} = C(\dot\phi\cos\theta + \dot\psi)$$

$$\frac{\partial T}{\partial \psi} = 0$$

$$\frac{\partial V}{\partial \psi} = 0$$

conservation of moment of momentum about $\underline{k}$

$$\therefore \boxed{C(\dot\phi\cos\theta + \dot\psi) = h \text{, a constant}} \quad \boxed{1}$$

ϕ: $$\frac{\partial T}{\partial \dot\phi} = A\sin^2\theta\,\dot\phi + C(\dot\phi\cos\theta + \dot\psi)\cos\theta$$

$$\frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial V}{\partial \phi} = 0$$

conservation of moment of momentum about $\underline{K}$

$$\therefore \boxed{A\sin^2\theta\,\dot\phi + h\cos\theta = H \text{, a constant}} \quad \boxed{2}$$

θ: $\dfrac{\partial T}{\partial \dot\theta} = A\dot\theta$

$$\frac{\partial T}{\partial \theta} = A\dot\phi^2 \sin\theta\cos\theta - C\dot\phi\sin\theta(\dot\phi\cos\theta + \dot\psi)$$
$$= A\dot\phi^2 \sin\theta\cos\theta - h\dot\phi\sin\theta$$

$$\frac{\partial V}{\partial \theta} = -mga\sin\theta$$

$$\therefore \boxed{A\ddot\theta - A\dot\phi^2\sin\theta\cos\theta + h\dot\phi\sin\theta - mga\sin\theta = 0} \quad \boxed{3}$$

c/ Steady precession, θ = constant
$\dot\phi$ = constant
$\dot\psi$ = constant

Use $\boxed{3}$

$$\therefore A\dot\phi^2\cos\theta - h\dot\phi + mga = 0$$
$$\therefore A\dot\phi^2\cos\theta - C(\dot\phi\cos\theta + \dot\psi)\dot\phi + mga = 0 \quad \boxed{4}$$
$$\therefore (A-C)\cos\theta\,\dot\phi^2 - C\dot\psi\dot\phi + mga = 0$$
$$\therefore \dot\phi = \frac{C\dot\psi \pm \sqrt{(C\dot\psi)^2 - 4(A-C)mga\cos\theta}}{2(A-C)\cos\theta} \quad \boxed{5}$$

For fast spin (ie $\dot\psi$ large) $\boxed{1}$ gives $h \simeq C\dot\psi$

Then $\boxed{4}$ gives $-C\dot\psi\dot\phi + mga \simeq 0$
(assuming $\dot\phi \ll \dot\psi$ – ie slow precession)

$$\therefore \boxed{\dot\phi \simeq \frac{mga}{C\dot\psi}}$$

The second solution use $\boxed{4}$ but this time with $\dot{\phi}$ comparable with $\dot{\psi}$

$\therefore$ neglect mga

$$\therefore A\dot{\phi}\cos\theta \approx C(\dot{\phi}\cos\theta + \dot{\psi})$$

$$\therefore \dot{\phi}(A-C)\cos\theta = C\dot{\psi}$$

$$\therefore \boxed{\dot{\phi} = \frac{C\dot{\psi}}{(A-C)\cos\theta}}$$

Alternatively use $\boxed{5}$ NOTE $\sqrt{1+\varepsilon} \simeq 1 + \frac{1}{2}\varepsilon$

$$\text{So } \dot{\phi} = \frac{C\dot{\psi} \pm C\dot{\psi}\sqrt{1 - \frac{4(A-C)\cos\theta\, mga}{(C\dot{\psi})^2}}}{2(A-C)\cos\theta}$$

The "−" solution gives $\dot{\phi} \simeq \dfrac{C\dot{\psi}\,\frac{1}{2}\,\frac{4(A-C)\cos\theta\, mga}{(C\dot{\psi})^2}}{2(A-C)\cos\theta}$

$$\simeq \frac{mga}{C\dot{\psi}} \quad *$$

The "+" solution gives $\dot{\phi} \simeq \dfrac{2C\dot{\psi}}{2(A-C)\cos\theta}$

$$= \frac{C\dot{\psi}}{(A-C)\cos\theta} \quad *$$

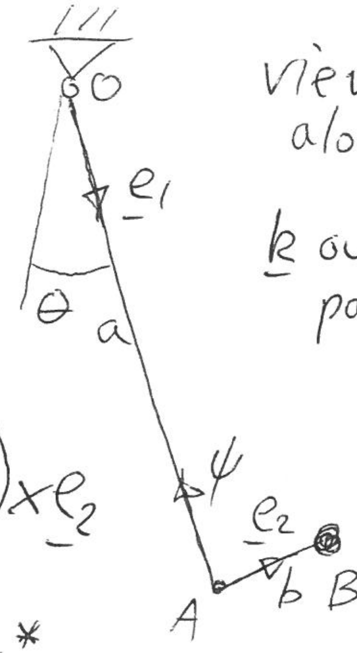
AS BEFORE

NOTE: For $\dot{\psi} \gg \dot{\phi}$ then $\dot{\psi} \simeq \omega_3$

So $\dot{\phi} \simeq \dfrac{mga}{C\omega_3}$ as well

but for $\dot{\phi}$ of same order as $\dot{\psi}$ then use $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ to give $\dot{\phi} = \dfrac{C\omega_3}{A\cos\theta}$

4/ The hard bit is to get the velocity of B. Hence the "show that" for T.



view along $\underline{k}$

$\underline{k}$ out of page

$$\underline{r}_B = a\,\underline{e}_1 + b\,\underline{e}_2$$

$$\underline{v}_B = a(\dot\theta\underline{k})\times\underline{e}_1 + b(\dot\theta\underline{k} - \dot\psi\underline{e}_1)\times\underline{e}_2$$

Define $\underline{k}\times\underline{e}_1 = \underline{e}_1^*$ which is in the plane of paper

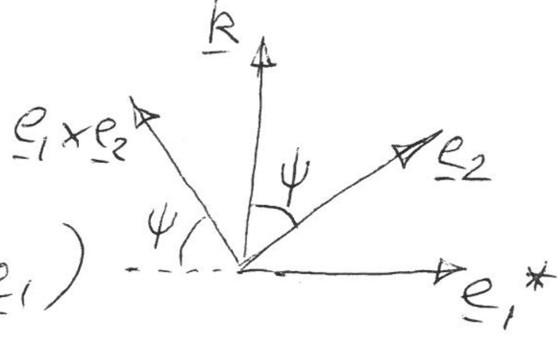
(note AB isn't in the plane of paper)

<u>view along $\underline{e}_1$</u>

$\underline{e}_1$ out of page

Resolve $\underset{\sim}{v}_B$ into $\underline{e}_1$ $\underline{e}_1^*$ $\underline{k}$ directions

(note $\underline{k}\times\underline{e}_2 = -\sin\psi\,\underline{e}_1$)



$$\underset{\sim}{v}_B = a\dot\theta\,\underline{e}_1^* + b\dot\theta(-\sin\psi\,\underline{e}_1) - b\dot\psi(\underline{e}_1\times\underline{e}_2)$$

note $\underline{e}_1\times\underline{e}_2 = -\cos\psi\,\underline{e}_1^* + \sin\psi\,\underline{k}$

$$\therefore v_B = (a\dot\theta + b\dot\psi\cos\psi)\underline{e}_1^* - b\dot\theta\sin\psi\,\underline{e}_1 - b\dot\psi\sin\psi\,\underline{k}$$

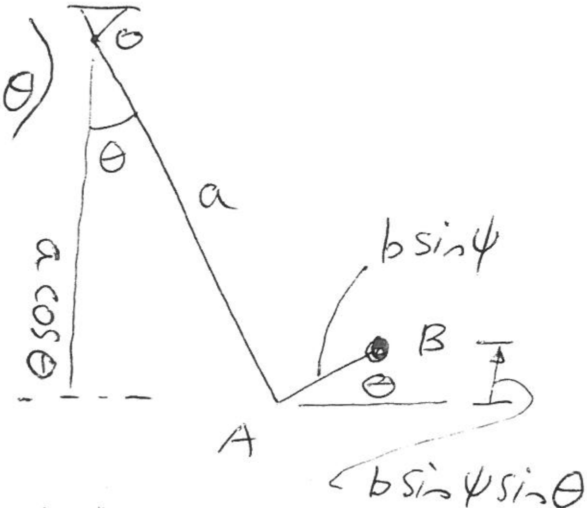
so $T = \frac{1}{2}m\left((a\dot\theta + b\dot\psi\cos\psi)^2 + (b\dot\theta\sin\psi)^2 + (b\dot\psi\sin\psi)^2\right)$

$$= \frac{1}{2}m\left(a^2\dot\theta^2 + 2ab\dot\theta\dot\psi\cos\psi + b^2\dot\psi^2\cos^2\psi + b^2\dot\theta^2\sin^2\psi + b^2\dot\psi^2\sin^2\psi\right)$$

$$= \frac{1}{2}m\left(a^2\dot\theta^2 + b^2\dot\psi^2 + b^2\dot\theta^2\sin^2\psi + 2ab\dot\theta\dot\psi\cos\psi\right)$$

and

$V = mg\left(a(1-\cos\theta) + b\sin\psi\sin\theta\right)$



θ:

$$\frac{\partial T}{\partial\dot\theta} = m\left(a^2 + b^2\sin^2\psi\right)\dot\theta + mab\dot\psi\cos\psi$$

$$\frac{\partial T}{\partial\theta} = 0 \qquad \frac{\partial V}{\partial\theta} = mg\left(a\sin\theta + b\sin\psi\cos\theta\right)$$

$$\therefore \frac{d}{dt}\left[m(a^2+b^2\sin^2\psi)\dot\theta + mab\dot\psi\cos\psi\right] + mg\left(a\sin\theta + b\sin\psi\cos\theta\right) = 0$$

$$\therefore ma^2\ddot\theta + 2mb^2\dot\theta\dot\psi\sin\psi\cos\psi + mab\cos\psi\,\ddot\psi + mb^2\sin^2\psi\,\ddot\theta - mab\dot\psi^2\sin\psi + mga\sin\theta + mgb\sin\psi\cos\theta = 0$$

(This was important to get full marks because the "Show that" for ψ was too easy to fudge!)

ψ: $\frac{\partial T}{\partial \dot{\psi}} = mb^2\dot{\psi} + mab\dot{\theta}\cos\psi$

$$\frac{\partial T}{\partial \psi} = mb^2\dot{\theta}^2 \sin\psi\cos\psi - mab\dot{\theta}\dot{\psi}\sin\psi$$

$$\frac{\partial V}{\partial \psi} = mgb\cos\psi\sin\theta$$

$$\frac{d}{dt}\left[mb^2\dot{\psi} + mab\dot{\theta}\cos\psi\right] - mb^2\dot{\theta}^2\sin\psi\cos\psi + mab\dot{\theta}\dot{\psi}\sin\psi + mgb\cos\psi\sin\theta = 0$$

$$\therefore\ b\ddot{\psi} + a\ddot{\theta}\cos\psi - a\dot{\theta}\dot{\psi}\sin\psi - b\dot{\theta}^2\sin\psi\cos\psi + a\dot{\theta}\dot{\psi}\sin\psi + g\cos\psi\sin\theta = 0$$

$$\therefore\ \boxed{b\ddot{\psi} + a\ddot{\theta}\cos\psi - b\dot{\theta}^2\sin\psi\cos\psi + g\cos\psi\sin\theta = 0}$$

c/ linearize T & V and neglect small terms

$$T = \frac{1}{2}m\left(a^2\dot{\theta}^2 + b^2\dot{\psi}^2 + 2ab\dot{\theta}\dot{\psi}\right)$$

$$V = mg\left(\frac{1}{2}a\theta^2 + b\theta\psi\right)$$

$$\therefore\ [M] = m\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \text{ and } [K] = mg\begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$$

4/d/ For natural frequencies

$$|(K) - (M)\omega^2| = 0$$

$$\therefore \begin{vmatrix} mga - ma^2\omega^2 & mgb - mab\omega^2 \\ mgb - mab\omega^2 & 0 - mb^2\omega^2 \end{vmatrix} = 0$$

$$\therefore (ga - a^2\omega^2)(-b^2\omega^2) - (gb - ab\omega^2)^2 = 0$$

$$\therefore (g - a\omega^2)\, a\omega^2 + (g - a\omega^2)^2 = 0$$

$$\therefore (g - a\omega^2)(a\omega^2 + g - a\omega^2) = 0$$

$$\therefore g(g - a\omega^2) = 0$$

$$\therefore \omega^2 = \frac{g}{a}$$

There appears to be only one solution!