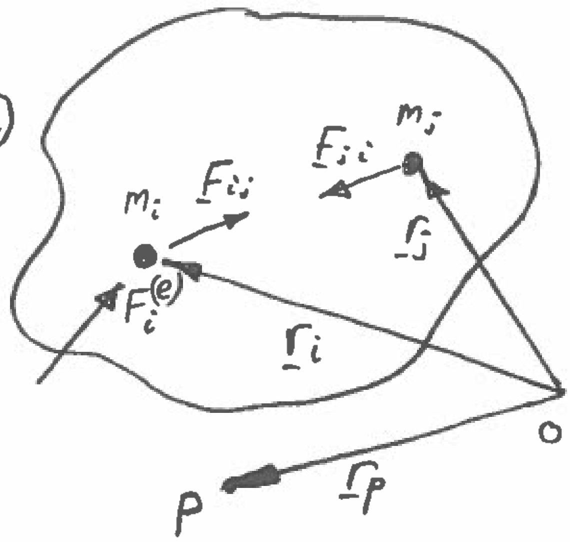
3C5 2019

(a) Newton's law for a particle : $\underline{F} = m\ddot{\underline{r}}$ ①

(i) A rigid body is a collection of particles m_i each obeying ①

$$\underline{F}_i^{(e)} + \sum_{j \neq i} \underline{F}_{ij} = m_i \ddot{\underline{r}}_i \quad ②$$

total external force acting on m_i ; sum of all internal forces between m_i and all other particles



Sum ② over all particles i

$$\therefore \sum_i \underline{F}_i^{(e)} + \underbrace{\sum_i \sum_{j \neq i} \underline{F}_{ij}}_{} = \sum m_i \ddot{\underline{r}}_i \quad ③$$

This term is zero because internal forces cancel in pairs by Newton III

Define : $\underline{F}^{(e)} = \sum_i \underline{F}_i^{(e)}$ = total external force

and $M = \sum m_i$ = total mass

and $\underline{r}_G$ = position of centre of mass

such that $M\underline{r}_G = \sum_i m_i \underline{r}_i$

and $\underline{p} = \sum_i m_i \dot{\underline{r}}_i$ = total linear momentum ④

$= M\dot{\underline{r}}_G$

$\therefore \dot{\underline{p}} = \sum_i m_i \ddot{\underline{r}}_i$

So that (3) becomes $\underline{F}^{(e)} = \dot{\underline{p}}$

(ii) Also define $\underline{h}_P = \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \dot{\underline{r}}_i)$

= the total moment of momentum of all particles about an arbitrary point P

$$\therefore \dot{\underline{h}}_P = \sum_i (\dot{\underline{r}}_i - \dot{\underline{r}}_P) \times m_i \dot{\underline{r}}_i + \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

and note that $\dot{\underline{r}}_i \times \dot{\underline{r}}_i = 0$, and also that $\dot{\underline{r}}_P$ can be taken out of the summation

$$\therefore \dot{\underline{h}}_P = -\dot{\underline{r}}_P \times \sum_i m_i \dot{\underline{r}}_i + \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

$$\therefore \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i) = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p} \qquad (5)$$

from (4)

Now take moments of (2) about P

$$\therefore (\underline{r}_i - \underline{r}_P) \times \underline{F}_i^{(e)} + \sum_{j \neq i} (\underline{r}_i - \underline{r}_P) \times \underline{F}_{ij} = (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

(6)

and sum over all particles m_i noting that

$\sum_i \sum_{j \neq i} (\underline{r}_i - \underline{r}_P) \times \underline{F}_{ij} = 0$ because the moments about P of all internal forces $\underline{F}_{ij}$ cancel in pairs

Also define $\underline{Q}_P^{(e)} = \sum_i (\underline{r}_i - \underline{r}_P) \times \underline{F}_i^{(e)}$ = the total moment of external forces about P

so that (6) becomes

$$\sum_i (\underline{r}_i - \underline{r}_P) \times \underline{F}_i^{(e)} + 0 = \sum_i (\underline{r}_i - \underline{r}_P) \times (m_i \ddot{\underline{r}}_i)$$

$$\therefore \quad \underline{Q}_P^{(e)} = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p} \qquad (7)$$

using (5)

b/ If P and G are coincident then

$$\dot{\underline{r}}_P = \dot{\underline{r}}_G \qquad \therefore \quad \dot{\underline{r}}_P \times \underline{p}$$

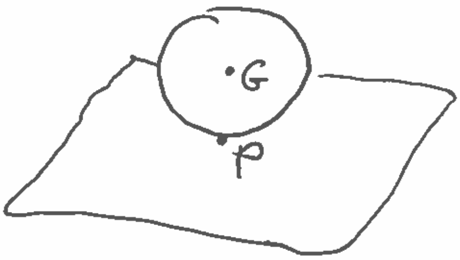
$$= \dot{\underline{r}}_G \times (m \dot{\underline{r}}_G)$$

$$= 0$$

$$\therefore \quad \underline{Q}_P^{(e)} = \dot{\underline{h}}_P$$

This is a special result, easier to understand than (7) ...

c/



No matter how the table moves, point P is always directly below G so that $\dot{\underline{r}}_P = \dot{\underline{r}}_G$

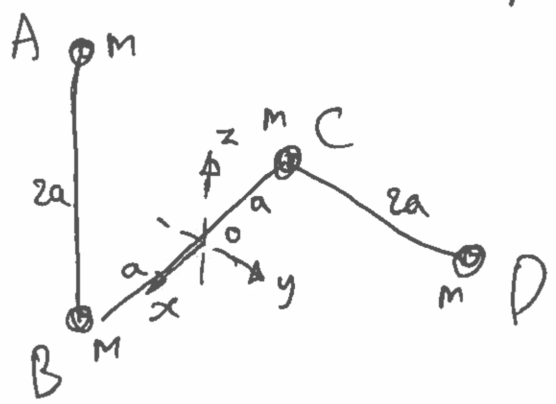
and since $\underline{p} = m \dot{\underline{r}}_G$

then $\dot{\underline{r}}_P \times \underline{p} = 0$

Note also that all forces acting on the ball pass through P $\quad \therefore \quad \underline{Q}_P = 0$ $\quad \therefore \quad \dot{\underline{h}}_P = 0$

$\therefore$ $\underline{h}_P$ is conserved.

2.



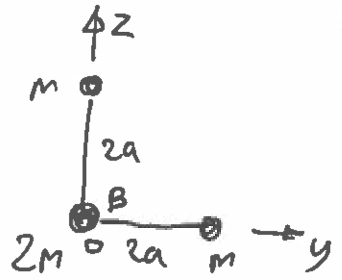
b/ $x_G = 0$

$y_G = \frac{a}{2}$

$z_G = \frac{a}{2}$

a/

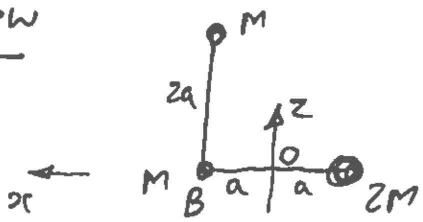
x-view



$I_{xx} = 2m(2a)^2 = 8ma^2$

$I_{yz} = 0$

y-view

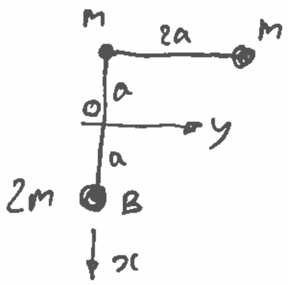


$I_{yy} = 2ma^2 + ma^2 + m(a^2+4a^2)$

$= 8ma^2$

$I_{xz} = 2ma^2$

z-view



$I_{zz} = 8ma^2$ (as I_{yy})

$I_{xy} = -2ma^2$

$$\therefore I_O = ma^2 \begin{bmatrix} 8 & 2 & -2 \\ 2 & 8 & 0 \\ -2 & 0 & 8 \end{bmatrix}$$

c/ for I_G use parallel axis theorem

$$I_O = I_G + 4m \begin{bmatrix} Y^2+Z^2 & -XY & -XZ \\ -YX & X^2+Z^2 & -YZ \\ -ZX & -ZY & X^2+Y^2 \end{bmatrix}$$

with (x, y, z) being the position of O relative to G

$\therefore \quad x = 0 \quad y = -\frac{a}{2} \quad z = -\frac{a}{2}$

$$\therefore \quad I_G = ma^2 \begin{bmatrix} 8 & 2 & -2 \\ 2 & 8 & 0 \\ -2 & 0 & 8 \end{bmatrix} - 4ma^2 \begin{bmatrix} \frac{1}{4}+\frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= ma^2 \begin{bmatrix} 6 & 2 & -2 \\ 2 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}$$

d/ Two ways to get I_B

i/ First principles, use views on previous page

x-view $\quad I_{xx} = 8ma^2$

$I_{yz} = 0$

y-view $\quad I_{yy} = m(2a)^2 + 2m(2a)^2 = 12ma^2$

$I_{xz} = 0$

z-view $\quad I_{zz} = m(2a)^2 + m\left((2a)^2 + (2a)^2\right)$

$= 12\,ma^2$

$I_{xy} = -4ma^2$

$$\therefore \quad I_B = 4ma^2 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

ii/ check by alternative method which is parallel axis from G

$$\underline{r}_{B/G} = \underline{r}_B - \underline{r}_G = [a, 0, 0] - [0, \tfrac{a}{2}, \tfrac{a}{2}]$$

$$= \begin{array}{ccc} (a, & -\tfrac{a}{2}, & -\tfrac{a}{2}) \\ x & y & z \end{array}$$

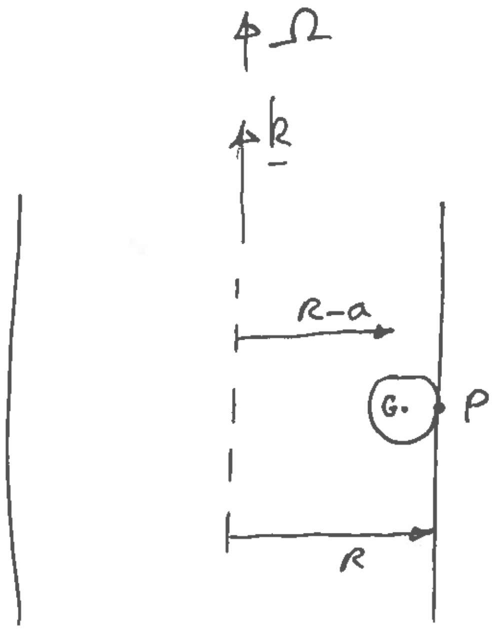
$$\therefore \quad I_B = ma^2 \begin{bmatrix} 6 & 2 & -2 \\ 2 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix} + 4ma^2 \begin{bmatrix} \frac{1}{4}+\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1+\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & 1+\frac{1}{4} \end{bmatrix}$$

$$= ma^2 \begin{bmatrix} 8 & 4 & 0 \\ 4 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad \text{as before}$$

e/ I_B has a principal axis apparent with zeros in 3rd column & row

$\therefore$ a principal moment of inertia is $12ma^2$

3 (a)



$\underline{j}$ into page

Ball is an AAA body $A = \frac{2}{5}Ma^2$

body angular velocity

$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$

reference frame angular velocity

$\underline{\Omega} = \Omega \underline{k}$

a/ No slip at P

$$\underline{\dot{r}}_P = \underline{\dot{r}}_G + (\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}) \times a\underline{i}$$

$$= (R-a)\Omega\underline{j} + a\omega_3\underline{j} - a\omega_2\underline{k} = 0$$

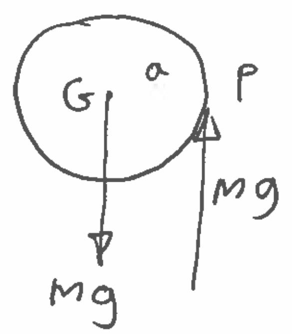
$$\therefore \quad \omega_3 = \frac{(R-a)}{a}\Omega = \frac{v_G}{a}$$

$$\omega_2 = 0 \quad /// \qquad = \frac{v_P}{a}\frac{R-a}{R} \quad ///$$

ω_1 unconstrained

note
$v_G = (R-a)\Omega$
$v_P = R\Omega$

b/



In steady state vertical force at P = Mg so use

$\underline{Q} = \underline{\dot{h}}$ at G

$$\therefore \; -Mga\underline{j} = \Omega\underline{k} \times \underline{h}$$

$$= \Omega \underline{k} \times \left(A\left(\omega_1 \underline{i} + \frac{R-a}{a}\Omega\underline{k}\right)\right)$$

$$= \Omega A \omega_1 \underline{j}$$

$$\therefore \quad \omega_1 = \frac{-Mga}{A\Omega} = \frac{-5MgaR}{2Ma^2 v_P}$$

$$= \frac{-5}{2}\frac{gR}{a v_P} \quad ///$$

3(b)

~~[illegible]~~ At a specified time t, knowledge of q_i does not give knowledge of $\dot{q}_i$ - For example to compute the motion from a set of initial conditions both q_i and $\dot{q}_i$ must be specified. Together q_i and $\dot{q}_i$ form the state vector of the system and Lagrange considers variations of the system in state space $\Rightarrow$ q_i and $\dot{q}_i$ must be considered to be independent variables [25%]

(b) Example: large angle motions of a double pendulum.

Lagrange
$$\underbrace{\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right]}_{\text{chain rule}} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$\sum_j \left\{ \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \frac{d}{dt}(\dot{q}_j) + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j} \frac{d}{dt}(q_j) \right\}$$

$$\Rightarrow \sum_j \left\{ \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j} \dot{q}_j \right\} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

not a damping term, since for $Q_i = 0$ we know $\frac{d}{dt}[T+U] = 0$

[25%]

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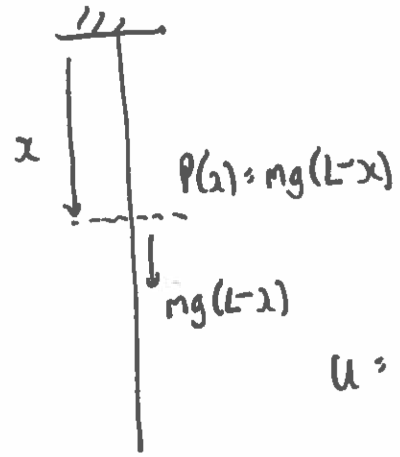
4(a)



$$w(x,t) = q_1(x/L) + q_2(x/L)^2$$

$$w' = q_1/L + 2q_2(x/L^2)$$

$$\Rightarrow U = \tfrac{1}{2}(mg/L^2)\int_0^L (L-x)\left[q_1 + 2q_2(x/L)\right]^2 dx$$

$$U = \tfrac{1}{2}(mg/L^2)\int_0^L (L-x)\left[\underbrace{q_1^2}_{q_1^2(1-\frac{1}{2})L^2} + \underbrace{4q_1q_2(x/L)}_{4q_1q_2(\frac{1}{2}-\frac{1}{3})L^2} + \underbrace{4q_2^2(x/L)^2}_{4q_2^2(\frac{1}{3}-\frac{1}{4})L^2}\right]dx$$

$$\underline{U = \tfrac{1}{2}(mg/L)\left[\tfrac{1}{2}q_1^2 + \tfrac{2}{3}q_1q_2 + \tfrac{1}{3}q_2^2\right]}$$

$$T = \tfrac{1}{2}m\int_0^L \dot{w}^2 dx = \tfrac{1}{2}m\int_0^L\left[\dot{q}_1(x/L)^2 + 2\dot{q}_1\dot{q}_2(x/L)^3 + \dot{q}_2^2(x/L)^4\right]dx$$

$$\underline{T = \tfrac{1}{2}m\left[\tfrac{1}{3}\dot{q}_1^2 + \tfrac{1}{2}\dot{q}_1\dot{q}_2 + \tfrac{1}{5}\dot{q}_2^2\right]}$$

Lagrange $\Rightarrow$
$$\begin{pmatrix} \frac{\partial^2 T}{\partial \dot{q}_1^2} & \frac{\partial^2 T}{\partial \dot{q}_1\partial \dot{q}_2} \\ \frac{\partial^2 T}{\partial \dot{q}_1\partial \dot{q}_2} & \frac{\partial^2 T}{\partial \dot{q}_2^2} \end{pmatrix}\begin{pmatrix}\ddot{q}_1\\ \ddot{q}_2\end{pmatrix} + \begin{pmatrix} \frac{\partial^2 U}{\partial q_1^2} & \frac{\partial^2 U}{\partial q_1\partial q_2} \\ \frac{\partial^2 U}{\partial q_1\partial q_2} & \frac{\partial^2 U}{\partial q_2^2} \end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$$

$$\underline{\Rightarrow m\begin{pmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{pmatrix}\begin{pmatrix}\ddot{q}_1\\ \ddot{q}_2\end{pmatrix} + (mg/L)\begin{pmatrix} 1/2 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}}$$

[35%]

$Q_1 = Fq_1$; $Q_2 = Fq_2$ from $\delta W = Q\delta q$

(b) For $q_2 = 0 \Rightarrow \frac{1}{3}m\ddot{q}_1 + (mg/L)\frac{1}{2}q_1 = 0$

Put $q_1 = \theta L$ and multiply by L^2

$$\Rightarrow \frac{1}{3}ML^3\ddot{\theta} + \frac{1}{2}MgL^2\theta = 0$$

$$\Rightarrow \underline{I_0\ddot{\theta} + \frac{1}{2}MLg\theta = 0}$$

[10%]

$$\omega_n = \sqrt{\frac{3}{2}}\sqrt{g/L} = \underline{1.225\sqrt{g/L}}$$

(c) For $q_1 = 0$ $\quad \frac{1}{5}m\ddot{q}_2 + \frac{1}{3}(mg/L)q_2 = 0$

$$\omega_n = \sqrt{5/3}\sqrt{g/L} = \underline{1.291\sqrt{g/L}}$$

(d+e)

$$\begin{pmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \underbrace{(g/L)}_{\alpha}\begin{pmatrix} 1/2 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(-\frac{1}{3}\omega^2 + \frac{1}{2}\alpha\right)\left(-\frac{1}{5}\omega^2 + \frac{1}{3}\alpha\right) - \left(-\frac{1}{4}\omega^2 + \frac{1}{3}\alpha\right)^2 = 0$$

$$\omega^4\left(\frac{1}{15} - \frac{1}{16}\right) - \alpha\omega^2\left(\frac{1}{10} + \frac{1}{9} - \frac{1}{6}\right) + \alpha^2\left(\frac{1}{6} - \frac{1}{9}\right) = 0$$

$$0.0042\,\omega^4 - \alpha\omega^2(0.0444) + \alpha^2\,0.0556 = 0$$

$$\omega^2 = \left(0.0444\alpha \pm \alpha\sqrt{0.0444^2 - 4 \times 0.0042 \times 0.0556}\right) \times \left(\frac{1}{2 \times 0.0042}\right)$$

$$\omega^2 = \begin{matrix} 1.45\alpha \\ 9.1199\alpha \end{matrix} \Rightarrow \left.\begin{matrix} \omega_1 = 1.2042\sqrt{g/L} \\ \omega_2 = 3.019\sqrt{g/L} \end{matrix}\right\} \text{vs } \underline{\text{exact}} \quad \begin{matrix} 1.202\sqrt{g/L} \\ 2.76\sqrt{g/L} \end{matrix}$$

<u>Add more terms to improve</u>