3C6 Solutions 2015

1 (a) From Data Sheet, equation is $\rho J \frac{\partial^2\theta}{\partial t^2} = GJ\frac{\partial^2\theta}{\partial x^2}$ for free motion.

At a fixed end, say $x=0$, $\theta = 0$

At a free end, $x=L$, torque is zero. Torque $= GJ\frac{\partial\theta}{\partial x}$

so the condition is $\frac{\partial\theta}{\partial x} = 0$

For a mode, put $\theta(x,t) = u(x)\,e^{i\omega t}$

Equation requires $Gu'' + \rho\omega^2 u = 0$

or $u'' + k^2 u = 0$, $k^2 = \frac{\omega^2}{c^2}$, $c^2 = \frac{G}{\rho}$

General solution is $u = A\cos kx + B\sin kx$

$u(0) = 0 \rightarrow A = 0$

$u'(L) = 0 \rightarrow kB\cos kL = 0$

$\therefore \cos kL = 0$, so $k = \frac{(n-\frac{1}{2})\pi}{L}$ and $\omega = \frac{c(n-\frac{1}{2})\pi}{L}$

for $n = 1, 2, 3 \dots$

Mode 1

, $\omega_1 = \frac{c\pi}{2L}$

Mode 2

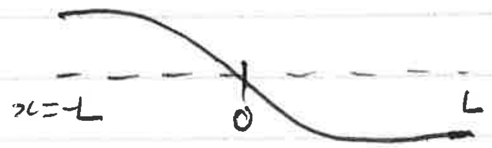
$\omega_2 = \frac{3c\pi}{2L}$

Mode 3

$\omega_3 = \frac{5c\pi}{2L}$

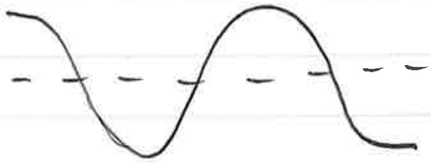
(b) For free-free shaft, need $u' = 0$ at both ends, but shapes are still sinusoidal. First mode has u = constant: rigid rotation at $\omega = 0$

Mode 2:



Mode 3:

Mode 4:



Modes are alternately symmetric and antisymmetric

(c) For equilibrium under equal and opposite end torques, $\theta(x)$ is simply linear:

By symmetry, there is a node at the centre. The subsequent motion must remain antisymmetric, so it can only involve modes 2, 4, 6 etc of the shaft.
For $0 \le x \le L$, these are identical to the modes from (a), so the transient response is the same as the response of (a) to a suddenly released force, copied antisymmetrically into $-L \le x \le 0$.
The input is a step, but a downward step from a steady value to zero. So we can use the step response formula, but each modal response will go as $\cos \omega_n t$, not $(1-\cos\omega_n t)$ as for the upward step.
To use the formula, first normalise the modes from (a).
Need $u_n(x) = K_n \sin \dfrac{(n-\frac{1}{2})\pi x}{L}$ such that $\int_0^L \rho J u_n^2\, dx = 1$

$$\therefore \rho J K_n^2 \underbrace{\int_0^L \sin^2 \frac{(n-\frac{1}{2})\pi x}{L}\, dx}_{= L/2} = 1$$

$$\therefore K_n^2 = \frac{2}{\rho J L} \quad \text{with} \quad J = \frac{\pi a^4}{2}$$

So required step response is

$$\theta(x,t) = \underset{\uparrow}{\theta_0(x,0)} - Q \sum_n \frac{u_n(x)\, u_n(L)}{\omega_n^2}\left(1 - \cos\omega_n t\right)$$

Initial static shape

If we had small damping, the term in $\cos\omega_n t$ would decay away to zero as $t \to \infty$. But we know that $\theta(x,t) \to 0$ as $t \to \infty$, so the other term in the summation must cancel the initial shape $\theta_0(x,0)$

So step response is

$$\theta(x,t) = Q \sum_n \frac{2}{\rho J L} \frac{\sin(n-\frac{1}{2})\pi \, \sin\frac{(n-\frac{1}{2})\pi x}{L}}{\omega_n^2} \cos\omega_n t$$

$$= Q \cdot \frac{2}{\rho J L} \cdot \frac{L^2}{c^2\pi^2} \sum_n \frac{(-1)^{n+1}}{(n-\frac{1}{2})^2} \sin\frac{(n-\frac{1}{2})\pi x}{L} \cos\omega_n t$$

with $\omega_n = \frac{c\pi}{L}(n-\frac{1}{2})$ and $J = \frac{\pi a^4}{2}$

$$= \frac{4LQ}{G a^4 \pi^3} \sum_n \frac{(-1)^{n+1}}{(n-\frac{1}{2})^2} \sin\frac{(n-\frac{1}{2})\pi x}{L} \cos\frac{c\pi(n-\frac{1}{2})t}{L}$$

2 (a) From Data Sheet, equation is $\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$

for free motion.

At a pinned end, $w = 0$ (no displacement) and

$EI \frac{\partial^2 w}{\partial x^2} = 0$ (no bending moment)

For a mode, $w = u(x) e^{i\omega t}$

so equation requires $\frac{d^4 u}{dx^4} = \alpha^4 u$, $\alpha^4 = \frac{\rho A}{EI} \omega^2$

General solution is $u = K_1 \cos \alpha x + K_2 \sin \alpha x + K_3 \cosh \alpha x + K_4 \sinh \alpha x$

At $x = 0$: $\begin{cases} u = 0 \to K_1 + K_3 = 0 \\ u'' = 0 \to K_1 - K_3 = 0 \end{cases}$

$\therefore K_1 = K_3 = 0$

At $x = L$: $\begin{cases} u = 0 \to K_2 \sin \alpha L + K_4 \sinh \alpha L = 0 \\ u'' = 0 \to -\alpha^2 K_2 \sin \alpha L + \alpha^2 K_4 \sinh \alpha L = 0 \end{cases}$

So $K_2 \sin \alpha L$ <u>and</u> $K_4 \sinh \alpha L$ both $= 0$

$\sinh \alpha L = 0$ only for $\alpha = 0$: not interesting here as this would mean $w = 0$, and a pinned-pinned beam has no rigid-body modes. So must have $K_4 = 0$

Can't now have $K_2 = 0$ or there would be nothing left in $u(x)$, so must have $\sin \alpha L = 0$

$\therefore \alpha L = n\pi$, $n = 1, 2, 3 \cdots$

$$\therefore \omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

and mode shapes $u_n(x) = \sin \frac{n\pi x}{L}$

(b) Any continuous system can be approximated by a series of discrete models, with finer and finer resolution. This could be done by some kind of mass-spring sequence of models, or e.g. by a series of Finite-Element models with finer and finer mesh size. Mode orthogonality has been proved for all these discrete systems, and so it remains true in the limiting case of the continuous system.
The discrete orthogonality condition is $\underline{u}_n^t M \underline{u}_m = 0$ for $n \neq m$. This amounts to taking the expression for the kinetic energy, and substituting two different mode shapes for the two versions of the vector $\dot{\underline{y}}$.
So do the same for the continuous system: kinetic energy is $\frac{1}{2}\int \frac{\partial w}{\partial t}^2 dm$

and so orthogonality requires $\int u_n(x)\, u_m(x)\, dm = 0$

for $n \neq m$
For the beam from (a), $u_n = \sin\frac{n\pi x}{L}$ and $dm = \rho A dx$

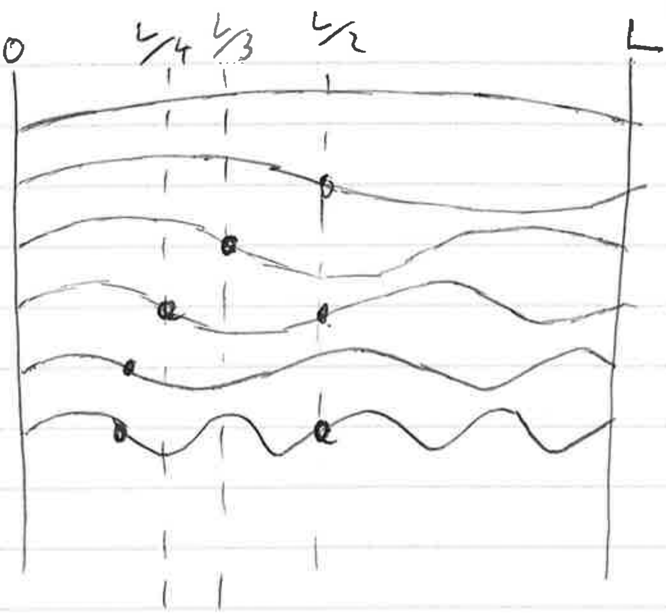
$$\text{So } \int u_n u_m \, dx = \rho A \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx$$

$$= \rho A \int_0^L \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right] dx$$

$$= \frac{\rho A}{2}\frac{L}{\pi}\left[-\frac{\sin\frac{(n-m)\pi x}{L}}{n-m} + \frac{\sin\frac{(n+m)\pi x}{L}}{n+m}\right]_0^L = 0 \text{ if } n \neq m$$

(or quote orthogonality result for Fourier series)

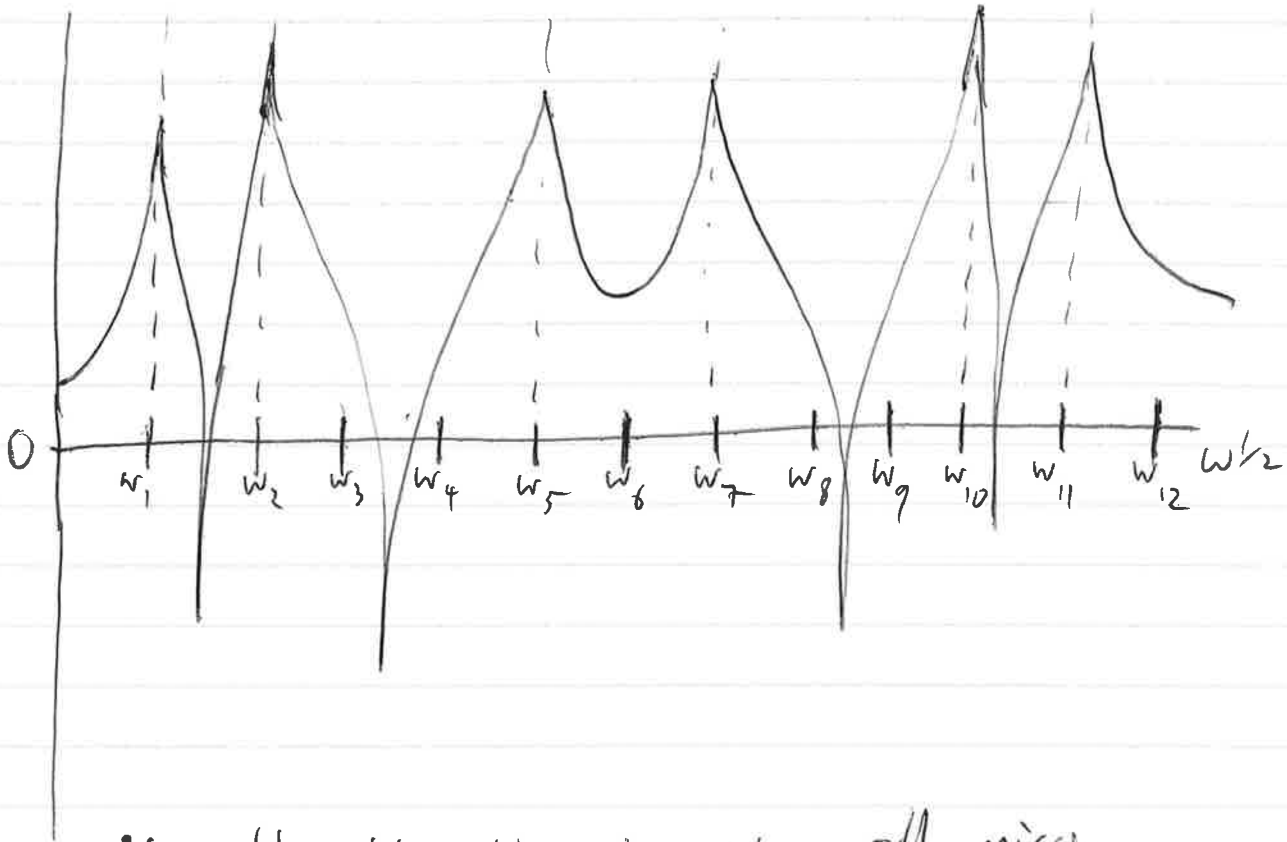
(c) First few modes:



Data we need:

n	$\sin n\pi/3$	$\sin n\pi/4$	product	Antiresonance?
1	+ ·866	+ ·707	+ ·612	yes
2	+ ·866	+1	+ ·866	
3	0	+ ·707	0	yes
4	−·866	0	0	
5	−·866	−·707	+·612	No
6	0	−1	0	
7	+·866	−·707	−·612	yes
8	+·866	0	0	
9	0	+·707	0	
10	−·866	+1	−·866	yes
11	−·866	+·707	−·612	
12	0	0	0	

Since $w_n \propto n^2$, convenient to plot against $w^{1/2}$ so that resonances are equally spaced.

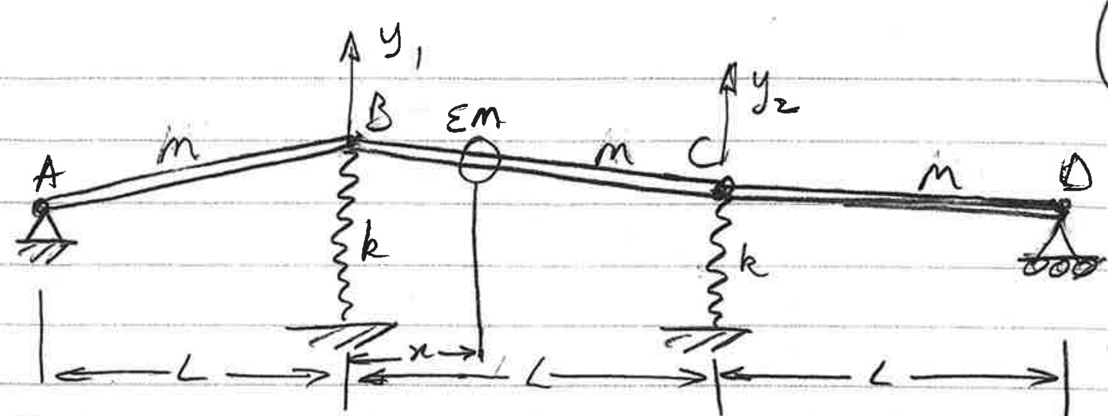


w_3, w_4, w_6, w_8, w_9, w_{12} all missing

w_2, w_{10} slightly higher peaks than the others

Pattern of antiresonances following the table

3



(a) $\varepsilon = 0$

$$KE: \; T = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left(\frac{\dot{y}_1}{L}\right)^2 + \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left(\frac{\dot{y}_2}{L}\right)^2$$

$$+ \frac{1}{2}m\left(\frac{\dot{y}_1+\dot{y}_2}{2}\right)^2 + \frac{1}{2}\left(\frac{mL^2}{12}\right)\left(\frac{\dot{y}_2-\dot{y}_1}{L}\right)^2$$

$$= \frac{1}{2}m\left\{\dot{y}_1^2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{12}\right) + \dot{y}_1\dot{y}_2\left(\frac{1}{2}-\frac{1}{6}\right) + \dot{y}_2^2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{12}\right)\right\}$$

$$= \frac{1}{2}m\left\{\frac{2}{3}\dot{y}_1^2 + \frac{1}{3}\dot{y}_1\dot{y}_2 + \frac{2}{3}\dot{y}_2^2\right\}$$

$$[M] = m\begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix}$$

$$PE: \; V = \frac{1}{2}ky_1^2 + \frac{1}{2}ky_2^2$$

$$[K] = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) EVP $\left([K]-\omega^2[M]\right)\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} k-2\omega^2m/3 & -\omega^2m/6 \\ -\omega^2m/6 & k-2\omega^2m/3 \end{bmatrix}\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = 0 \qquad \text{——— (1)}$$

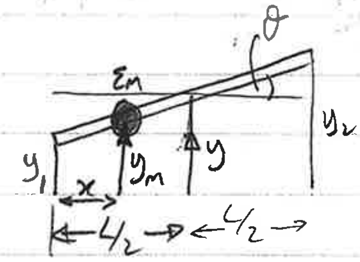
By inspection, mode shapes are $[1 \; 1]^T$ & $[1 \; -1]^T$

for $[1, 1]^T$: first row of (1) gives $k - \frac{2\omega^2 m}{3} - \omega^2 m/6 = 0 \Rightarrow \underline{\omega_1^2 = \frac{6}{5}\,k/m}$

for $[1 \; -1]^T$: first row of (1) gives $k - \frac{2}{3}\omega^2 m + \omega^2 m/6 = 0 \Rightarrow \underline{\omega_2^2 = 2k/m}$

(c) For the case when $\varepsilon \neq 0$ assume mode shapes are unchanged and use Rayleigh:

V is unchanged, but T is increased by $\Delta T = \frac{1}{2}(\varepsilon m)\dot{y}_m^2$



$\dot{y}_m = \dot{y} - (L/2 - x)\dot{\theta}$ with $\dot{y} = \frac{\dot{y}_1 + \dot{y}_2}{2}$

& $\dot{\theta} = \frac{\dot{y}_2 - \dot{y}_1}{L}$

For mode 1, $Y_1 = Y_2$ & $\theta = 0$ So $\Delta T = \frac{1}{2}(\varepsilon m)\dot{y}^2$

$= \frac{1}{2}(\varepsilon m)\left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2$

So $\omega_1^2 \simeq \frac{V_{max}}{T^*} = \frac{\cancel{\frac{1}{2}} k(1^2 + 1^2)}{\cancel{\frac{1}{2}} m(2/3 + 1/3 + 2/3) + \cancel{\frac{1}{2}}\varepsilon m}$

$= \frac{k}{m}\left(\frac{2}{5/3 + \varepsilon}\right) = \frac{k}{m}\left(\frac{6}{5 + 3\varepsilon}\right)$

For mode 2, $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$: $Y_1 = -Y_2$ & $\dot{y} = 0$ So $\Delta T = \frac{1}{2}(\varepsilon m)\left[\left(\frac{L}{2} - x\right)\left(\frac{\dot{y}_2 - \dot{y}_1}{L}\right)\right]^2$

$\omega_2^2 = \frac{V_{max}}{T^*} = \frac{\cancel{\frac{1}{2}} k(1^2 + 1^2)}{\cancel{\frac{1}{2}} m(2/3 - 1/3 + 2/3) + \cancel{\frac{1}{2}}(\varepsilon m)\left[\left(\frac{L}{2} - x\right)\left(\frac{2}{L}\right)\right]^2}$

$= \frac{k}{m}\left(\frac{2}{1 + \varepsilon\left[1 - \frac{2x}{L}\right]^2}\right)$

$\simeq \frac{k}{m}\Big(\underbrace{2}_{\text{old}} \underbrace{- 2\varepsilon(1 - 2x/L)^2}_{\text{shift due to } \varepsilon m}\Big)$ check $x = L/2$ $\omega_2^2 = 2k/m$ ✓

(d) Shift in lower mode is independent of x, so the size of the difference in frequency between the two modes depends only on ω_2

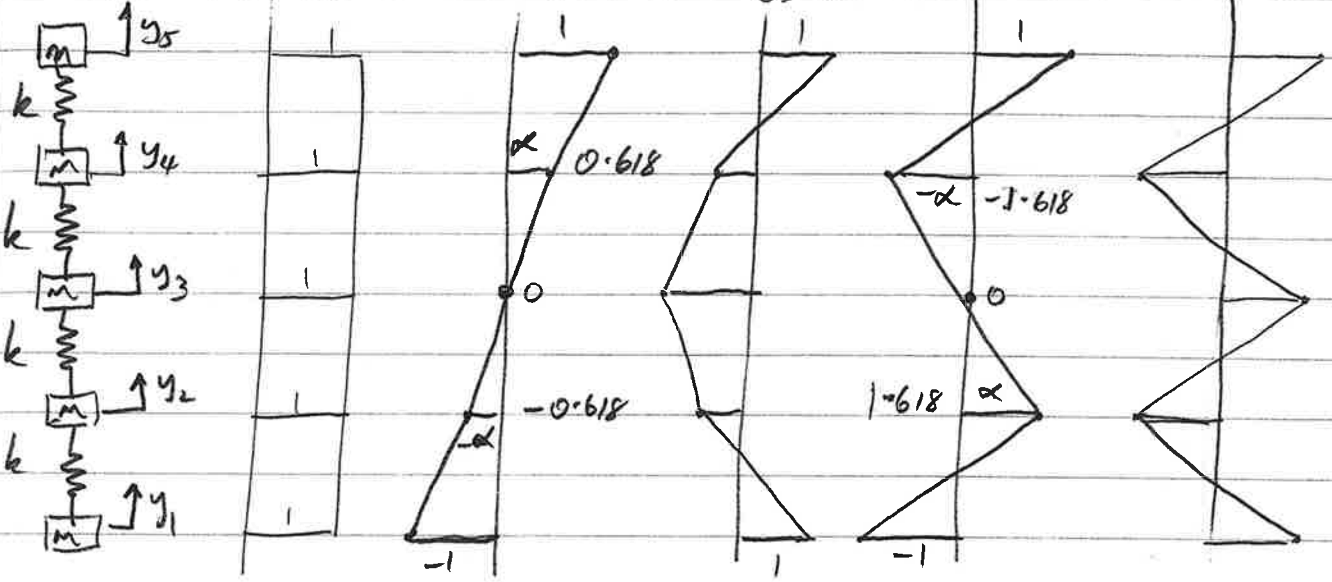
(i) Largest difference is when ω_2 is highest ie $x = L/2$ & ω_2 is unchanged from value in (b)

(ii) Smallest difference is when ω_2 is smallest ie $x = 0$ or $x = L$

At these positions, the pitch moment of inertia of the inner beam is largest. —2—

4.

$$V = \tfrac{1}{2}k\left[(y_2-y_1)^2+(y_3-y_2)^2+(y_4-y_3)^2+(y_5-y_4)^2\right]$$
$$T = \tfrac{1}{2}m\left[\dot{y}_1^2+\dot{y}_2^2+\dot{y}_3^2+\dot{y}_4^2+\dot{y}_5^2\right]$$



$[1\,1\,1\,1\,1]^T$ $\quad [-1\ {-\alpha}\ 0\ \alpha\ 1]^T$ $\quad [-1\ \beta\ 0\ {-\beta}\ 1]^T$

$\omega_1^2=0$	ω_2^2	ω_3^2	ω_4^2	ω_5^2
Rigid body	Anti-symm 1-node	Symm 2-nodes	Anti-symm 3-nodes	Symm 4-nodes

Modes 2 & 4:

Anti-symmetric modes both have shape $[-1\ {-\alpha}\ 0\ \alpha\ 1]^T$

Find these using Rayleigh

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{k\left[(y_2-y_1)^2+(y_3-y_2)^2+ \text{---}\right]}{m\left[y_1^2+y_2^2+y_3^2+\text{---}\right]} \qquad \text{———} \ (1)$$

for modes 2 & 4:

$$\omega^2 = \frac{k}{m}\,\frac{\left[(\alpha-1)^2+\alpha^2+\alpha^2+(1-\alpha)^2\right]}{1^2+\alpha^2+\alpha^2+1^2} = \frac{k}{m}\,\frac{\left[\cancel{2}(\alpha-1)^2+\cancel{2}\alpha^2\right]}{\cancel{2}\alpha^2+\cancel{2}} \quad (2)$$

$$\frac{d\omega^2}{d\alpha}=0: \quad \frac{(\alpha^2+1)(2(\alpha-1)+2\alpha)-\left[(\alpha-1)^2+\alpha^2\right][2\alpha]}{(\alpha^2+1)^2}=0$$

$\boxed{\text{Guess } \alpha=0.5 \quad \omega_2^2 = 0.4\,k/m}$

$$\Rightarrow (\alpha^2+1)(4\alpha-2)-(2\alpha^2-2\alpha+1)(2\alpha)=0$$
$$\cancel{4\alpha^3}-2\alpha^2+4\alpha-2-\cancel{4\alpha^3}+4\alpha^2-2\alpha=0$$
$$2\alpha^2+2\alpha-2=0 \longrightarrow \alpha^2+\alpha-1=0$$
$$\alpha = \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2} \qquad \text{———} \ (3)$$

To find ω^2, Substitute from (3) into (2)

$$\alpha = \frac{-1+\sqrt{5}}{2} = 0.618 \qquad \omega^2 = \left(\frac{2\alpha^2-2\alpha+1}{\alpha^2+1}\right)\frac{k}{m} \qquad = \left(2 - \frac{2\alpha+1}{\alpha^2+1}\right) k/m$$

$$\alpha^2 = \frac{(-1+\sqrt{5})^2}{4} = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2}$$

$$\therefore \frac{\omega^2}{k/m} = \frac{(3-\sqrt{5}) - (\sqrt{5}-1) + 1}{\frac{3-\sqrt{5}}{2}+1} = \frac{2(5-2\sqrt{5})}{5-\sqrt{5}} = \frac{2(5-2\sqrt{5})(5+\sqrt{5})}{25-5}$$

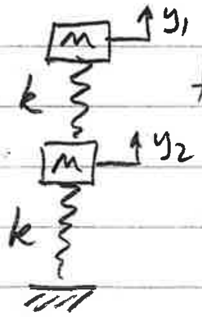
$= \frac{3-\sqrt{5}}{2} = \underline{\underline{0.382}}$ which is <u>less than</u> 0.4 from the guess $\alpha = 0.5$ as required ✓

for $\alpha = \frac{-1-\sqrt{5}}{2} = -1.618 \qquad \alpha^2 = \frac{3+\sqrt{5}}{2}$

$$\frac{\omega^2}{k/m} = 2 - \left[\frac{\cancel{(-1-\sqrt{5})} \cancel{+1}}{\frac{3+\sqrt{5}}{2}+1}\right] = 2 + \frac{2\sqrt{5}}{5+\sqrt{5}} \times \frac{5-\sqrt{5}}{5-\sqrt{5}}$$

$$= \frac{3+\sqrt{5}}{2} = \underline{\underline{2.618}}$$

<u>Check</u>: In the anti-symm modes there is a node in the middle ⇒ The system behaves as if 2 DoF:



for which $[K] = k\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{matrix}y_1\\ y_2\end{matrix}$ & $[M] = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

CE is $\left|[K] - \omega^2[M]\right| = 0$

ie $\begin{vmatrix} k-\omega^2 m & -k \\ -k & 2k-\omega^2 m\end{vmatrix} = 0$

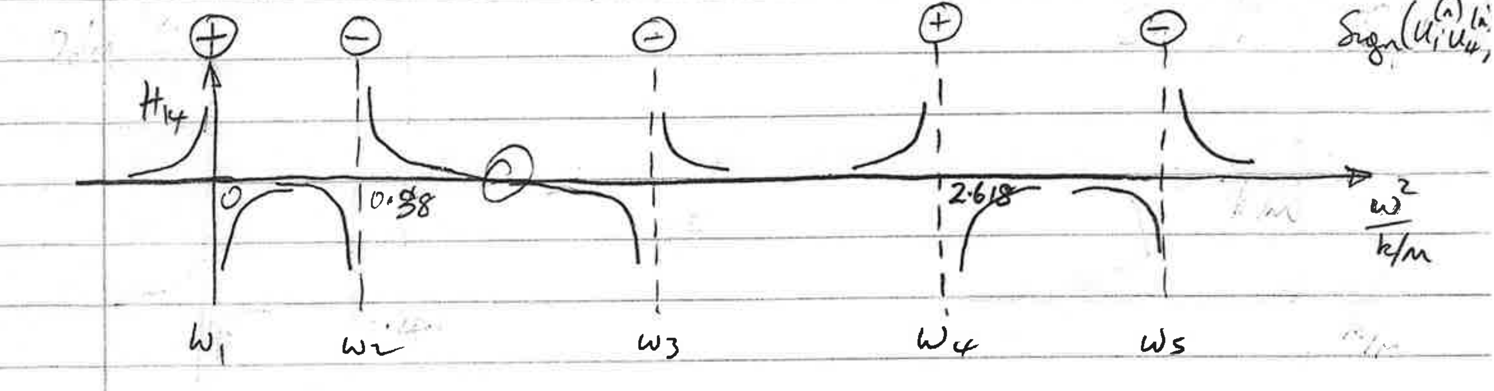
$$\Rightarrow (k-\omega^2 m)(2k-\omega^2 m) - k^2 = 0$$

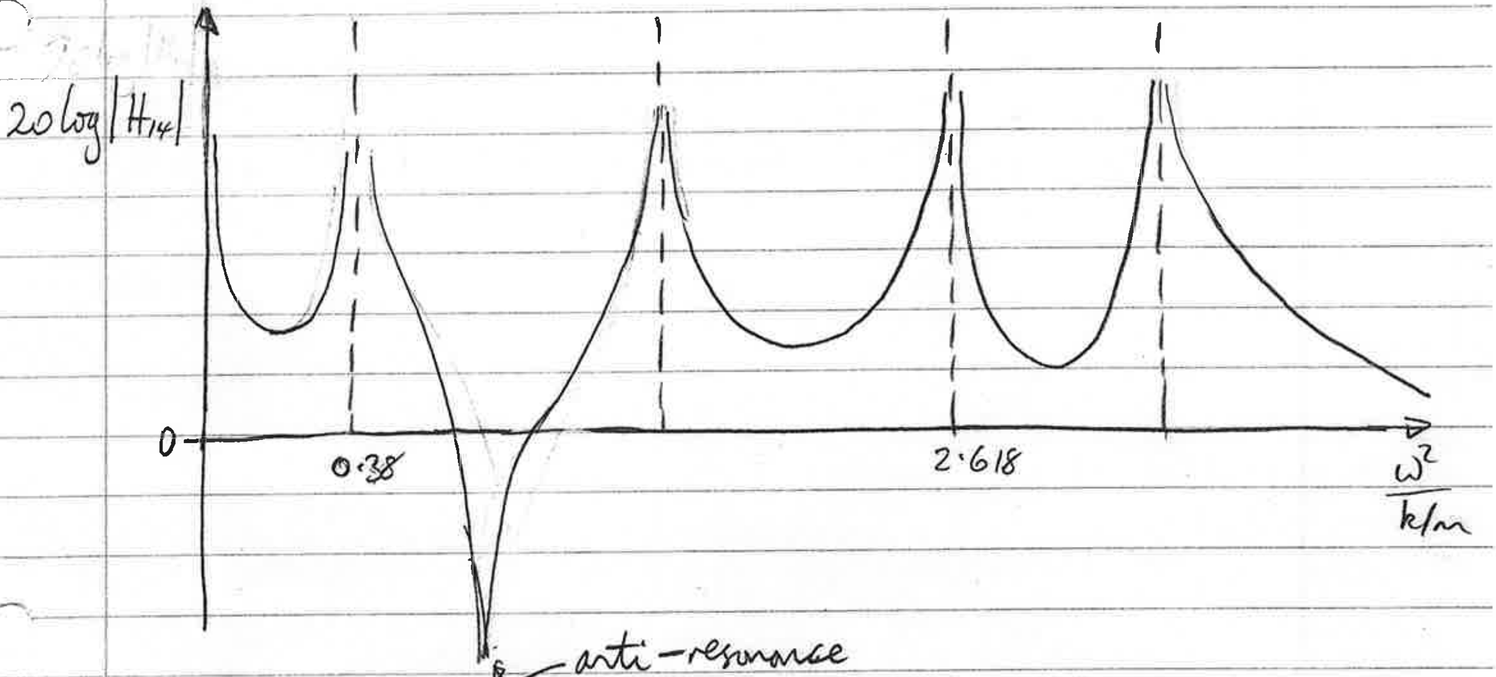
$$k^2 - 3k\omega^2 m + \omega^4 m^2 = 0 \Rightarrow \omega^4 - 3\frac{k}{m}\omega^2 + \frac{k^2}{m^2} = 0$$

$$\therefore \omega^2 = \frac{\frac{3k}{m} \pm \sqrt{9\frac{k^2}{m^2} - 4\frac{k^2}{m^2}}}{2} = \frac{k}{m}\left[\frac{3\pm\sqrt{5}}{2}\right] \checkmark$$

(c) Input at y_1, output at y_4

$$H_{14} = \frac{x_4}{F_1} = \sum_n \frac{u_4^{(n)} u_1^{(n)}}{\omega_n^2 - \omega^2}$$





(d) Mode 2 is anti-symmetric, so in mode 2, $y_5 = -y_1$
So when the column is excited in mode 2, the response at the top must have the same magnitude but be 180° out of phase
(NB No damping is needed to maintain a finite response for a displacement input.)

ASSESSOR'S COMMENTS
ENGINEERING TRIPOS PART IIA 2015
MODULE 3C6: VIBRATION

Q1 Torsional vibration

The least popular question, and several candidates tried to give answers relevant to bending beams rather than for torsion. The question is actually rather simple, based on the very start of the lecture course, but for some reason torsion is often thought to be difficult by students.

Q2 Beam vibration

Most did the bookwork of section (a) competently. Efforts on section (c) were very mixed. Many did not persevere to work out the pattern of signs for all the 12 modes asked for: they did the first few and then guessed, usually wrongly. But some did a good systematic job and were rewarded with high marks.

Q3 Hinged rods Rayleigh calculation

Most did section (a) well: those who got it wrong either forgot to include the rotational energy, or else used the wrong moment of inertia. Just one candidate failed to see the implications of the symmetry of the system for the two mode shapes in section (b). Many did the Rayleigh calculation well.

Q4 Four-DOF modal calculations and transfer function

Generally well done by most, with the curious exception that many did not read section (b) carefully enough to see that they were asked to *estimate* a frequency using Rayleigh (by guessing a suitable value of α), before continuing in section (c) to find the exact answer. The sketches for section (a) were very variable, some being extremely implausible! Many noticed that there was a rigid-body mode, but then failed to place it at zero frequency when doing section (d).

J Woodhouse (Principal Assessor)