Engineering Tripos Part IIB : Module 3C6 Solutions 2016

1(a). Potential energy $V = \frac{1}{2}k\left(x_1^2 + x_2^2 + x_3^2\right)$

Kinetic energy $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\left\{\left[b\dot{\theta}+\dot{x}_1\right]^2 + \left[b\dot{\theta}+\dot{x}_2\right]^2 + \left[b\dot{\theta}+\dot{x}_3\right]^2\right\}$

So stiffness matrix is $[K] = k\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Mass matrix is $[M] = \begin{bmatrix} I+3mb^2 & mb & mb & mb \\ mb & m & 0 & 0 \\ mb & 0 & m & 0 \\ mb & 0 & 0 & m \end{bmatrix}$ [20%]

(b) Three planes of mirror symmetry:

Modes are symmetric or antisymmetric in these planes. For symmetric motion,

e.g. try $\theta = 0$, $x_1 = 0$, $x_2 = -x_3 = y$

Then $T = m\dot{y}^2$, $V = ky^2$

so $\omega^2 = \frac{k}{m}$ //

It appears that there are 3 possible modes which all share this frequency: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

but notice that the third is a linear combination of the first two ((ii) − (i)).

In fact, at $\omega^2 = \frac{k}{m}$ there is a two-fold

1(b) cont. "degeneracy", and any linear combination of two of the above is a possible mode shape.

Next, there is obviously a rigid-body mode $\begin{bmatrix}\theta\\0\\0\\0\end{bmatrix}$ with frequency $\omega = 0$

One mode remains — it must be antisymmetric in all three planes ①, ②, ③. So it must be a rotationally symmetric shape $\begin{bmatrix}\theta\\x\\x\\x\end{bmatrix}$, and it must be orthogonal to the rigid mode

So
$$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}I+3mb^2 & mb & mb & mb\\ mb & m & 0 & 0\\ mb & 0 & m & 0\\ mb & 0 & 0 & m\end{bmatrix}\begin{bmatrix}\theta\\x\\x\\x\end{bmatrix} = 0$$

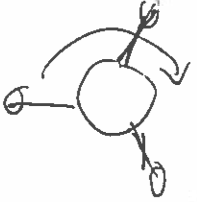
$\therefore \ (I+3mb^2)\theta + 3mbx = 0$, so $x = \left(b + \frac{I}{3mb}\right)\theta$

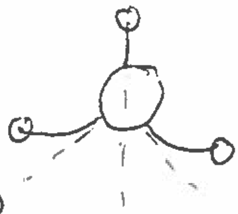
Then Rayleigh quotient gives $\omega^2 = \dfrac{k(I+3mb^2)}{mI}$ [60%]

(c) Now only one symmetry plane: say ①, assuming that the top spring is the different one.
2 modes unchanged: rigid rotation, and symmetric mode in ①

The other two must both be antisymmetric in plane ① and their frequencies must be higher because we have increased stiffness with no change in inertia

Unchanged:





Changed:

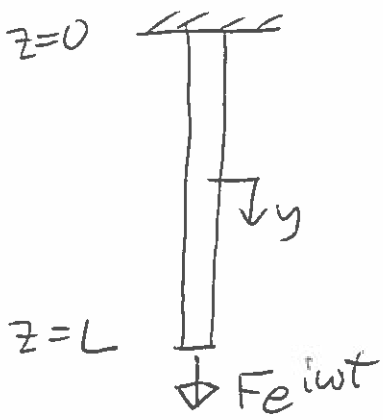
(was $\begin{bmatrix}0\\2\\-1\\-1\end{bmatrix}$)

[20%]

2(a) From data sheet, equation is

$$E\frac{\partial^2 y}{\partial z^2} - \rho\frac{\partial^2 y}{\partial t^2} = 0$$

or $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial z^2} \text{ with } c^2 = \frac{E}{\rho}$$



At $z=0$, $y=0$

At $z=L$, stress is $\frac{Fe^{i\omega t}}{A}$ so strain $\frac{\partial y}{\partial z} = \frac{Fe^{i\omega t}}{EA}$ [20%]

(b) Try $y(z,t) = Y(z)e^{i\omega t}$

Then $$c^2 Y'' = -\omega^2 Y$$

General solution is $Y = K_1 \cos\frac{\omega z}{c} + K_2 \sin\frac{\omega z}{c}$

At $z=0$, $Y=0 \rightarrow K_1 = 0$

At $z=L$, $Y' = \frac{F}{EA} \rightarrow K_2 \frac{\omega}{c}\cos\frac{\omega L}{c} = \frac{F}{EA}$

$$\therefore K_2 = \frac{Fc}{EA\omega \cos \omega L/c}$$

so $$G(L,z,\omega) = \frac{Y}{F} = \frac{c \sin \omega z/c}{EA\omega \cos \omega L/c}$$ [35%]

(c) For $z=L$, $$G(L,L,\omega) = \frac{c}{EA\omega}\tan\frac{\omega L}{c}$$

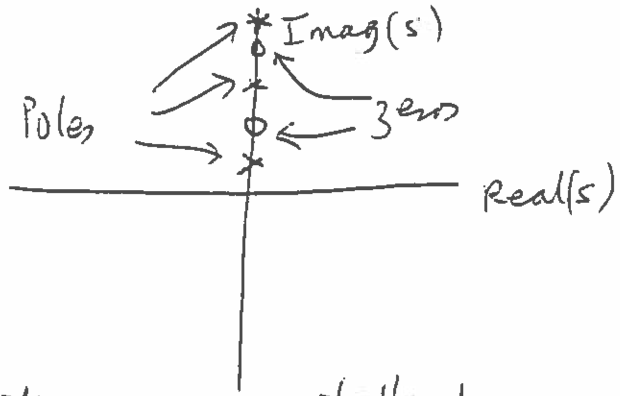
Poles where $\cos\frac{\omega L}{c} = 0 \rightarrow \frac{\omega L}{c} = (n-\tfrac{1}{2})\pi$, $n=1,2,3\ldots$

Despite appearances, ~~no~~ pole at $\omega=0$: when ω is small, $\tan\frac{\omega L}{c} \rightarrow \frac{\omega L}{c}$, so $G(L,L,\omega) \rightarrow \frac{L}{EA}$

Zeros where $\sin\frac{\omega L}{c} = 0 \rightarrow \frac{\omega L}{c} = n\pi$, $n=1,2,3$

Again, no zero at $\omega=0$

2(c) cont.



Zeros are the antiresonances of the bar, corresponding to frequencies where a very high axial force creates no (or little) displacement. [15%]

(d) Now apply displacement Y_0 at $z=L$, with $\omega = 3\Omega_0$

Dynamic force at $z=L$ is then $F = \dfrac{Y_0}{G(L, L, 3\Omega_0)}$

$$= \frac{Y_0 EA \cdot 3\Omega_0}{c \tan 3\Omega_0 L/c}$$

Drill bit will lift off if this amplitude exceeds the static pre-load W, so for smooth drilling we require

$$\frac{3 Y_0 EA \Omega_0}{c \tan 3\Omega_0 L/c} < W$$

Impossible to satisfy this near the antiresonance frequencies so must <u>avoid</u>

$$\frac{3\Omega_0 L}{c} \simeq n\pi, \quad n = 1, 2, 3 \ldots$$

ie avoid $\Omega_0 \simeq \dfrac{n\pi c}{3L}$, $n = 1, 2, 3 \ldots$

[30%]

5

3 (a) FRF shows low modal overlap throughout, with regularly spaced peaks with frequency ratios 1:3:5:7 etc. There is an antiresonance between every pair of peaks, and low values as frequency tends to zero, suggesting that there are no rigid-body modes. So it looks like a driving-point response of some system, and the regular peak spacing suggests that it is a system obeying the second-order wave equation: a string, or axial or torsional vibration of a bar. The frequency ratios suggest two possible configurations: (i) the response at the free end of a fixed-free system (column or torsion bar); or (ii) the response at the mid-point of a system with fixed boundaries at both ends: a string, or a fixed-fixed column or torsion bar. This ambiguity makes physical sense: if two fixed-free columns were joined together at the free ends, the result would be a double-length fixed-fixed column driven at the centre.

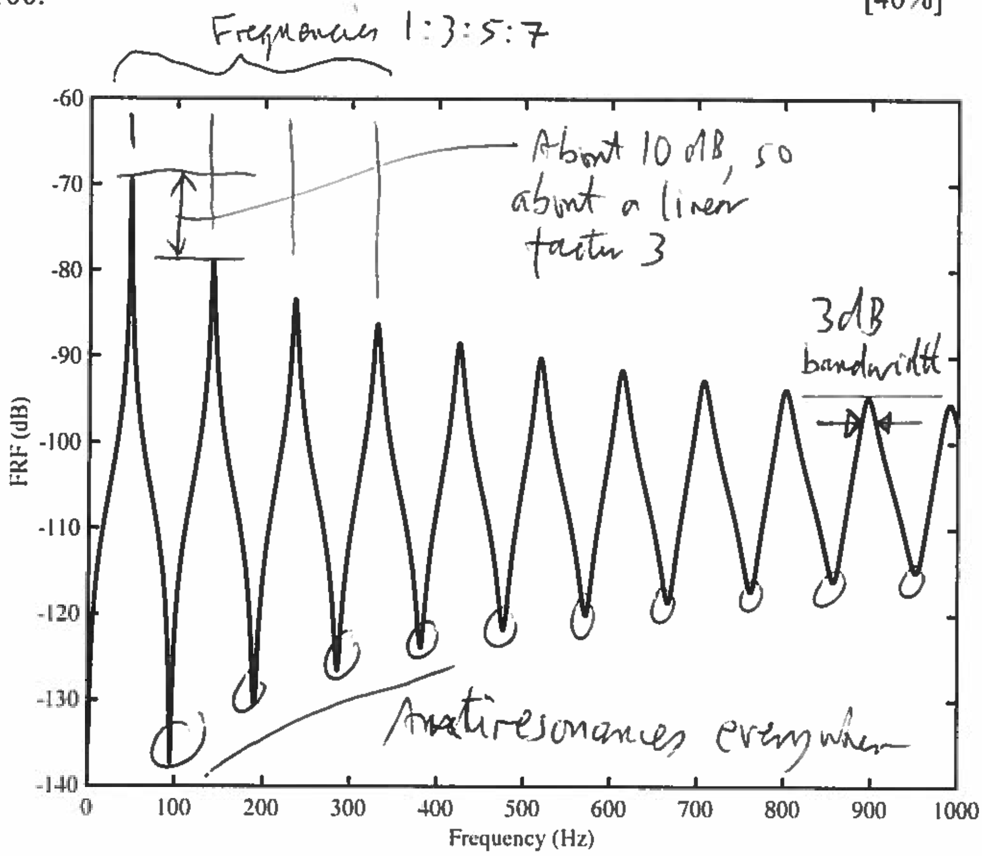
The standard modal formula for this FRF would read:

$$\frac{v_k}{f_j} \approx i\omega \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

so that near the nth peak,

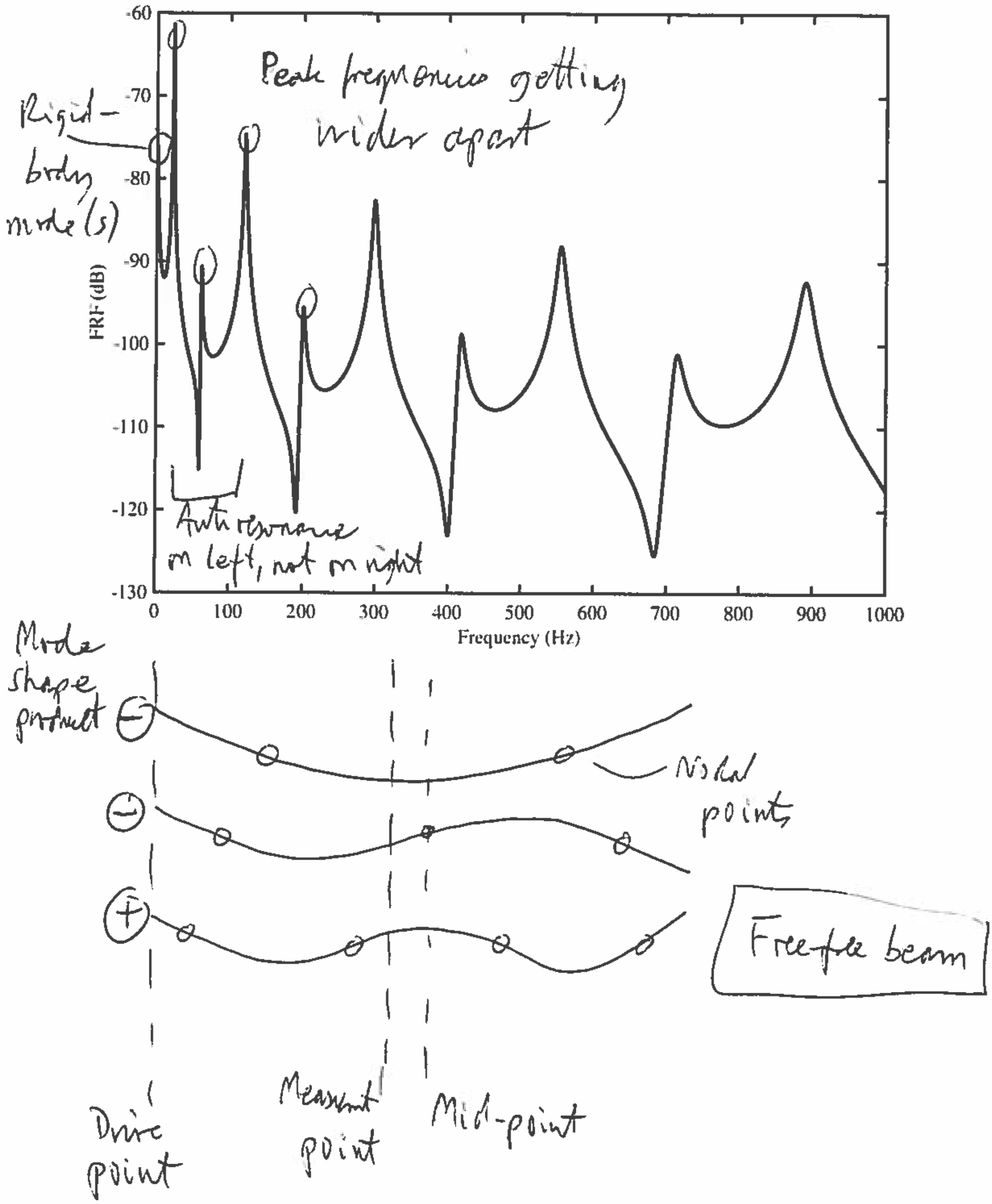
$$\frac{v_k}{f_j} \approx \frac{u_j^{(n)} u_k^{(n)}}{2i\omega\omega_n\zeta_n} = \frac{Q_n u_j^{(n)} u_k^{(n)}}{\omega_n}$$

The modes of the fixed-free column just described will be sinusoidal, and the normalised versions will all have similar magnitudes at the free end. So the peak height will be proportional to the Q factor and inversely proportional to the frequency. The peak heights reduce in a regular way, and simple measurement shows that it is consistent with heights (on a linear scale) inversely proportional to 1,3,5,7 etc. So the data suggests a constant Q factor for all modes. A 3 dB bandwidth can be estimated for the highest peaks, and gives a value for Q around 100. [40%]



3cont.

Low modal overlap again. Peak frequencies follow a regular pattern, but the get progressively wider apart: this suggests a bending beam of some kind, with natural frequencies following roughly as the square of the mode number. There is a peak at very low frequency, suggesting at least one rigid-body mode. All the even-numbered peaks have reduced heights compared to the odd-numbered ones, suggesting that either driving or observation is close to the middle of a symmetrical system. But this is clearly not a driving-point response because of the pattern of antiresonances. Apart from the even-off pattern, the peaks heights fall off in a regular way. A candidate system that satisfies all these conditions is a free-free bending beam, driven near one end and observed close to the centre. Not exactly AT the centre, because the even peaks are still visible. From the fact that the antiresonance is always to the left of each low peak, we can deduce that the observation point is displaced from the centre in the direction of the drive point: see sketch below. [40%]



3 (b) In a real measurement:

(i) peaks may not be precisely regular, because of effects like bending stiffness in a string, variations of properties along the length, or influence of boundary conditions;

(ii) damping would not be exactly independent of frequency;

(iii) peaks might be split into pairs for some kinds of system: for example vibration of a string or a bending beam can occur in two planes and they won't be exactly equal in practice;

(iv) a real system may have modes of more than one type (e.g. a beam can have bending, torsional and axial modes) so that the pattern of peaks will be more complicated;

(v) measurements will have some noise, so that antiresonances in particular will not be so clean;

(vi) measurements might show some electrical interference at multiples of 50 Hz (for UK electrical supply);

The last two points are unwanted effects, simply the result of imperfect measurements. The others all reflect true aspects of the physical system. Measurement noise can be reduced by averaging of the measurements, and by taking care over the quality of amplifiers etc. Electrical pickup can be minimised by good use of screening of cables, and care about the placement of signal cables relative to electrical supply cabling and power supplies of electrical systems.

[20%]

4(a)(i) From data sheet, $\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$

For a harmonic travelling wave $w = e^{i(\omega t - kx)}$,

require $-\rho A \omega^2 + EI k^4 = 0$

so $\omega = \sqrt{\frac{EI}{\rho A}}\, k^2$ or $k = \omega^{1/2}\left(\frac{\rho A}{EI}\right)^{1/4}$ = dispersion relation [10%]

(ii) For a mode, write $w(x,t) = Y(x)\, e^{i\omega t}$

Using k from the dispersion relation above, general solution is

$$Y = K_1 \cos kx + K_2 \sin kx + K_3 \cosh kx + K_4 \sinh kx$$

At $x=0$: $\begin{cases} Y = 0 \rightarrow K_1 + K_3 = 0 \\ Y' = 0 \rightarrow K_2 + K_4 = 0 \end{cases}$

At $x=L$: $\begin{cases} Y'' = 0 \rightarrow K_1(-\cos kL - \cosh kL) + K_2(-\sin kL - \sinh kL) = 0 \\ Y''' = 0 \rightarrow K_1(\sin kL - \sinh kL) + K_2(-\cos kL - \cosh kL) = 0 \end{cases}$

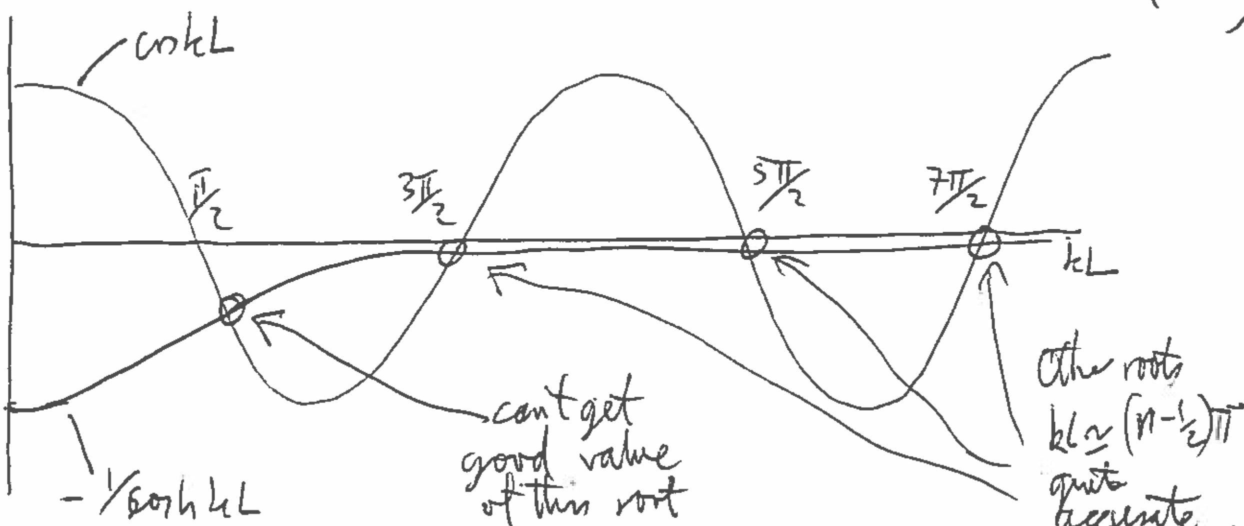
For non-trivial solution, 2×2 determinant of coefficients must vanish, so

$$(\cos kL + \cosh kL)^2 = -(\sin kL - \sinh kL)(\sin kL + \sinh kL)$$

$$\therefore \cos^2 kL + 2\cos kL \cosh kL + \cosh^2 kL = \sinh^2 kL - \sin^2 kL$$

$$\therefore \cos kL \cosh kL = -1, \text{ or } \cos kL = \frac{-1}{\cosh kL}$$ [30%]

(iii)



[20%]

4(b) Try to estimate first root by Rayleigh.

Try shape $U(x) = x^2$, so $U' = 2x$, $U'' = 2$

From data sheet, potential energy $V = \frac{1}{2} EI \int_0^L U''^2 dx$

$$= \frac{1}{2} EI \int_0^L 4\, dx = 2EIL$$

Reduced kinetic energy $T = \frac{1}{2} \rho A \int U^2 dx$

$$= \frac{1}{2} \rho A \int_0^L x^4 dx = \frac{1}{2} \rho A \cdot \frac{L^5}{5}$$

So Rayleigh quotient: $\omega^2 \simeq \frac{V}{T} = \frac{EI}{\rho A} \cdot \frac{20}{L^4}$ [30%]

(c) Impulse response from data sheet:

$$g(x, y, t) = \sum_{\text{modes } n} \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t$$

so for this case, $g(L, L, t) \simeq \sum_{n=1}^{\infty} \frac{U_L^2}{\omega_n} \sin \omega_n t$

Where $\omega_1 \simeq \sqrt{\frac{EI}{\rho A} \cdot \frac{20}{L^4}}$ from (b)

and $\omega_n \simeq \sqrt{\frac{EI}{\rho A}}\, k^2 = \sqrt{\frac{EI}{\rho A}} \left(\frac{(n - \frac{1}{2})\pi}{L} \right)^2$, $n = 2, 3, 4 \ldots$

from part (a) [10%]

3C6 2016 comments

Q1 Rotor, modes and frequencies of 4 degree-of-freedom system
This question related to material from the very earliest part of the course, and the performance of many candidates was rather disappointing. Many got confused between relative and absolute displacements, and thus did not get the mass and stiffness matrices correct. Despite the specific question about symmetry, very few discussed and exploited the symmetries of the system clearly, which really needs a sketch. More encouragingly, most succeeded in spotting at least some of the mode shapes, although not everyone noticed that there was a rigid-body mode.

Q2 Axial vibration of drillstring
Nearly all candidates used the mode-sum formula, forgetting what they had done on the first problem sheet to get a simpler closed-form answer to this problem. The mode-sum formula reveals the poles easily, but does not give the zeros: in any case, disappointingly few remembered that this driving-point response must have a zero between each pair of poles. Only one candidate answered part (d) correctly: the frequencies that must be avoided are the antiresonances, not the resonances.

Q3: Interpreting simulated measurements
Quite well done in parts. Everyone saw the regular peaks in the first example, and associated that with a string or axial vibration. But disappointingly few noticed the 1:3:5 pattern and its consequence for boundary conditions. No-one saw that the pattern of peak heights was consistent with constant damping for all modes, although a few did estimate a damping factor for a single peak.

Q4 Cantilever frequencies
The most popular question, and the most well-done. Many were a little hazy about the exact meaning of the dispersion relation, but almost everyone could derive the cantilever frequency equation and show the graphical solution. The Rayleigh section was also well done by most.