Question 1

PART IIA MODULE 3C6

EXAMINATION SOLUTIONS 2017

1 (a) A stretched string of length L has tension P and mass per unit length m. Both ends of the string are fixed, the distance from one end of the string is x and small amplitude transverse deflection of the string is denoted y.

(i) By solving the partial differential equation governing free small amplitude transverse vibration of the string, derive an expression for the mode shapes and natural frequencies of the string. [20%]

i) PDE: $$m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = 0$$

assume harmonic solution: $y = U(x)\, e^{i\omega t}$

$$\Rightarrow \quad m\omega^2 U + PU'' = 0.$$

so $U = A\sin kx + B\cos kx$ where $k = \omega\sqrt{m/P}$

or $\omega = k\sqrt{P/m}$

Boundary conditions: $U(0) = 0 \Rightarrow B = 0$

$U(L) = 0 \Rightarrow A\sin kL = 0.$

ie $\sin kL = 0$

$kL = n\pi$

$k = \frac{n\pi}{L}$, $\omega_n = \frac{n\pi}{L}\sqrt{P/m}$

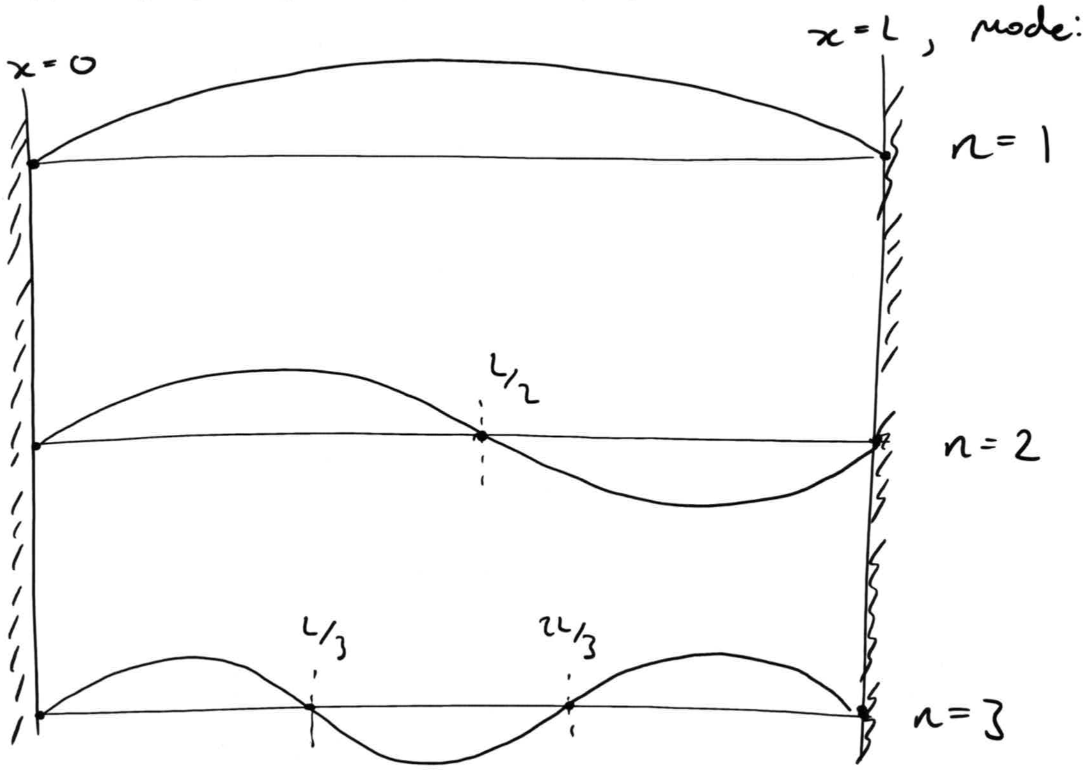
mode shapes $U_n = A\sin\frac{n\pi x}{L}$ //

natural frequencies: $\omega_n = \frac{n\pi}{L}\sqrt{P/m}$ //

(ii) Sketch the first three mode shapes of the string. [10%]

(iii) The string is driven at $x = x_1$ by a sinusoidal force of amplitude F at a frequency ω. Using your result from (a)(i) find an expression for the transfer function $G_a(x_1, x_2, \omega)$ from input force F to output transverse displacement y measured at an arbitrary position x_2 along the string. [20%]

ii)



iii) $$u(x) = A \sin \frac{n\pi x}{L}$$

mass normalise: $$\int u_n^2 \, dm = 1$$

$$\int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) . \, m \, dx = 1$$

$$\frac{mA^2}{2} \int_0^L \left(1 - \underbrace{\cos \frac{2n\pi x}{L}}_{\text{whole period so integral} = 0.}\right) dx = 1$$

$$\frac{mA^2}{2} \cdot L = 1 \quad , \quad A = \sqrt{2/mL}$$

$$G_a(x_1, x_3, \omega) = \sum_{n=1}^{\infty} \frac{u_n(x_1)\, u_n(x_3)}{\omega_n^2 - \omega^2}$$

$$G_a = \frac{2}{mL} \sum_{n=1}^{\infty} \frac{\left(\sin \frac{n\pi x_1}{L}\right)\left(\sin \frac{n\pi x_3}{L}\right)}{\left(\omega_n^2 - \omega^2\right)}$$

(b) Two strings of the same length L and mass per unit length m are now connected at $x = L/3$ from each of their ends by a light rigid strut, as illustrated in Figure 1. The tensions of the two strings are P_1 and P_2. For the case $P_1 = P_2$:

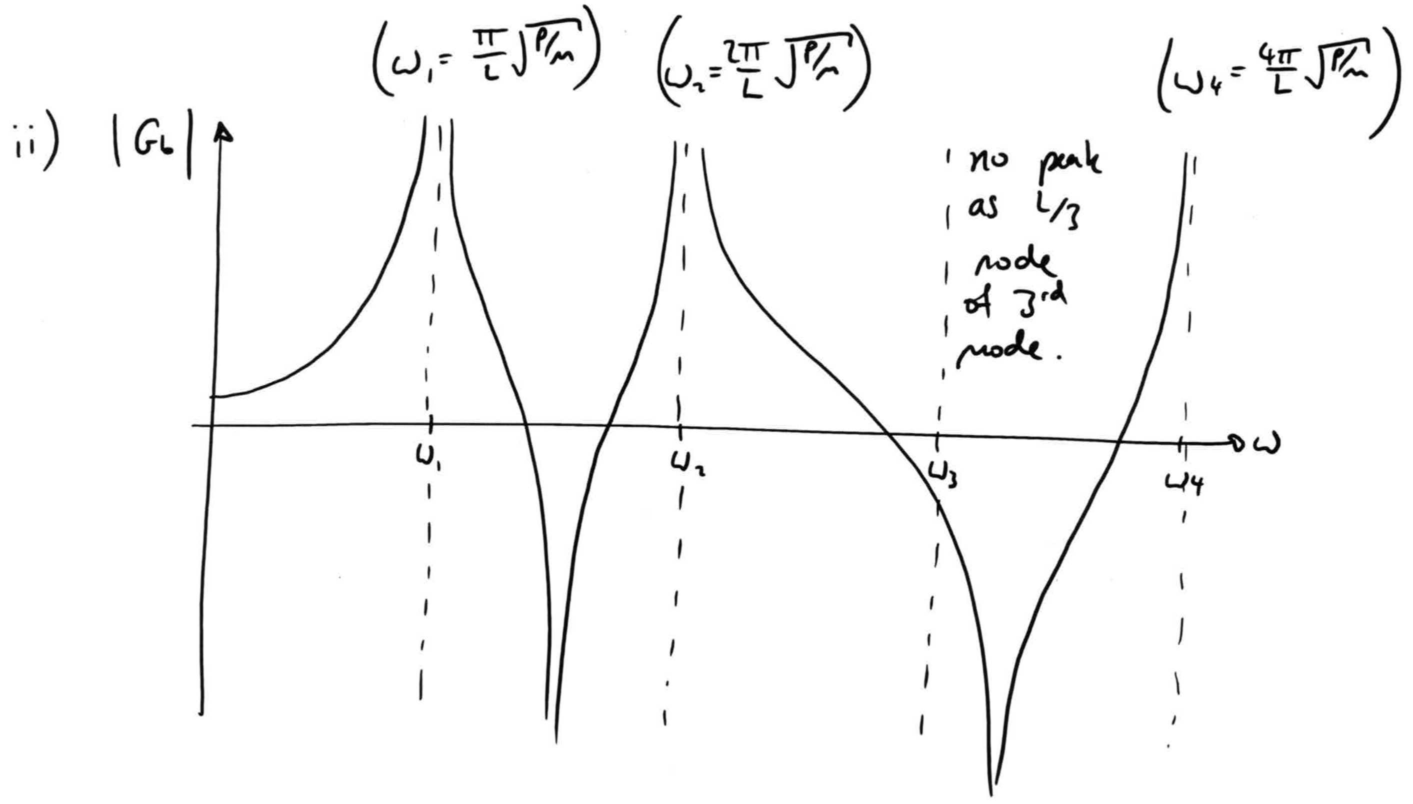
(i) What is the driving point transfer function G_b at the connection point of the coupled system? [10%]

(ii) Sketch the magnitude of G_b on a log-amplitude scale, including the first three peaks of the coupled system. [10%]

i) $$G_b = \left(\frac{1}{G_a} + \frac{1}{G_a}\right)^{-1} = \frac{G_a(L/3, L/3, \omega)}{2}$$

$$= \frac{1}{mL} \sum_{n=1}^{\infty} \frac{\left(\sin \frac{n\pi}{3}\right)^2}{\left(\omega_n^2 - \omega^2\right)}$$

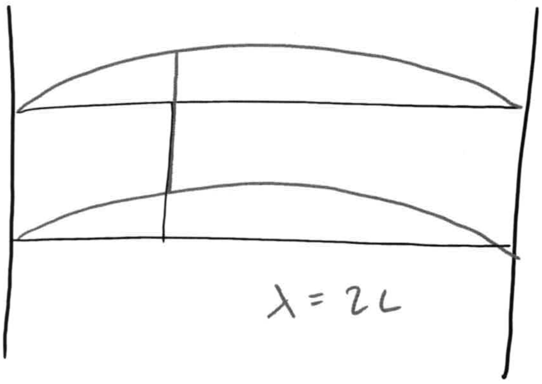
ii)



(iii) Sketch the first six mode shapes of the coupled system. Identify any mode shapes that correspond to peaks from your sketch of the transfer function G_b. [30%]

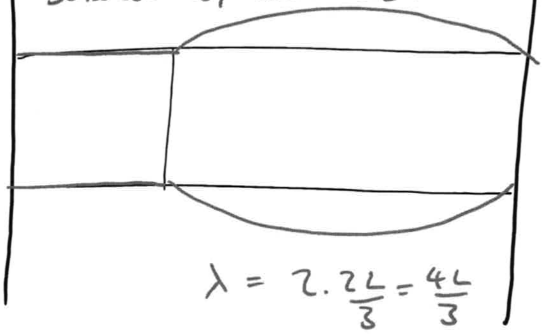
MODE 1

same as ω_1



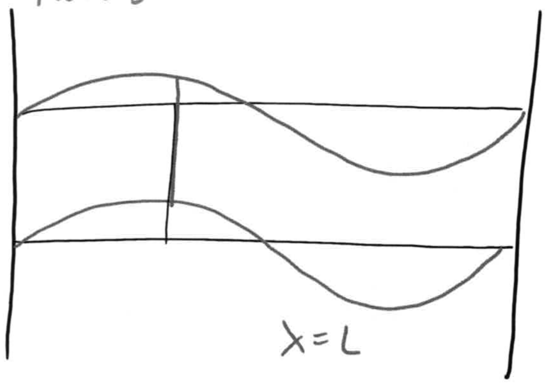
new frequency not calculated between ω_1 and ω_2.

MODE 2



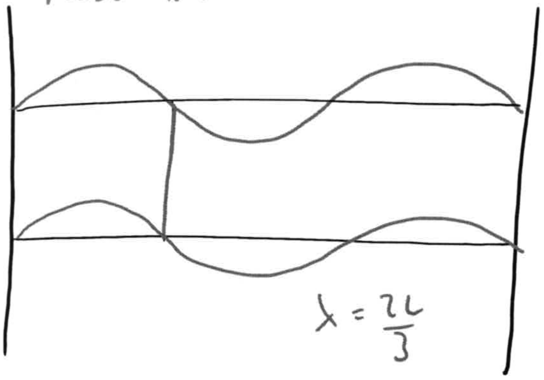
MODE 3

same as ω_2



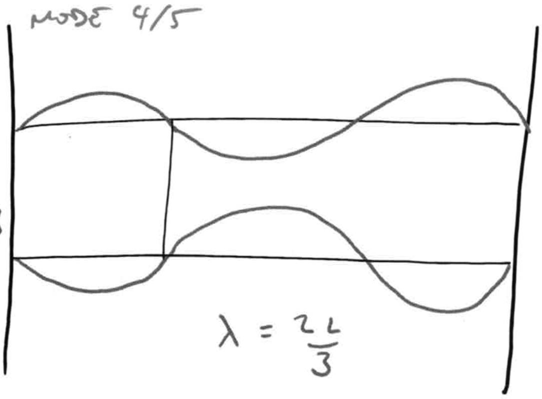
MODE 4/5

same as ω_3



MODE 4/5

ω_3

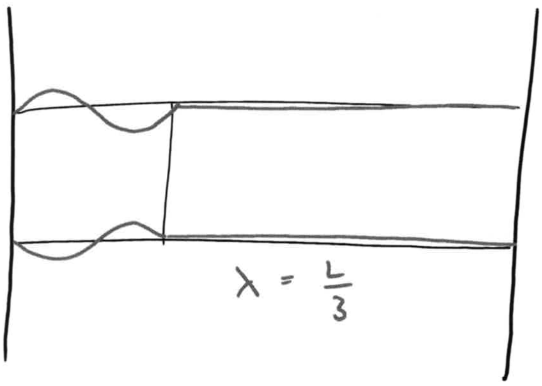


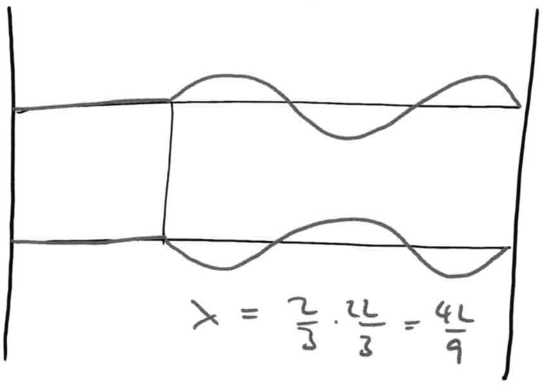
original modes, no relative motion @ $L/3$

modes with node @ $L/3$.

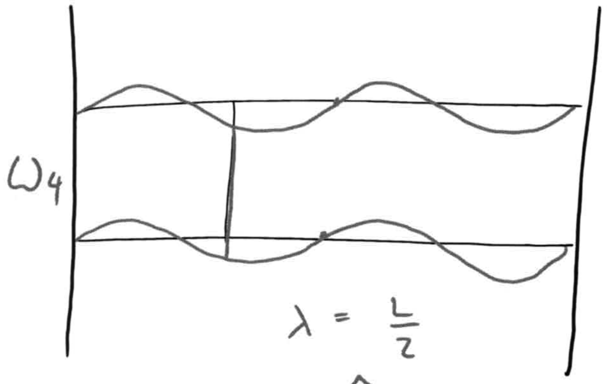
6^{th} mode needs bit more thought...

Candidates for 6th mode: need to check wavelengths to identify which is next in frequency sequence.





MODE 6



longest wavelength so this must be 6th frequency & mode shape

2 (a) A simple bridge is made using a uniform beam of length L, pinned at each end. The bridge has a solid rectangular cross section of width b and thickness d, with Young's Modulus E and density ρ.

(i) Derive an expression for the mode shapes and natural frequencies of the beam. [20%]

i) $EIy^{iv} + \rho A\ddot{y} = 0.$

assume harmonic solution: $y = U(x)e^{i\omega t}.$

$$EIU^{iv} - \rho A\omega^2 U = 0.$$

general solution:

$$U = A\sin kx + B\cos kx + C\sinh kx + D\cosh kx.$$

$$k^2 = \omega\sqrt{\rho A/EI}$$

$$\text{or } \omega = k^2\sqrt{EI/\rho A}$$

$$U = A\sin kx + B\cos kx + C\sinh kx + D\cosh kx$$

$$U' = k(A\cos kx - B\sin kx + C\cosh kx + D\sinh kx)$$

$$U'' = k^2(-A\sin kx - B\cos kx + C\sinh kx + D\cosh kx).$$

BC's: $U(0) = 0 \Rightarrow B + D = 0.$ no displacement

$U''(0) = 0. \Rightarrow -B + D = 0.$ no bending moment.

so $B = D = 0.$

$u(L) = 0 \Rightarrow \quad A \sin kL + C \sinh kL = 0.$

$u''(L) = 0 \rightarrow \quad -A \sin kL + C \sinh kL = 0.$

so $\quad A \sin kL = 0 \quad \& \quad C \sinh kL = 0$

$\Downarrow$

$C = 0.$

hence $\quad \sin kL = 0.$

$kL = n\pi$

$k = \frac{n\pi}{L}$ as per stretched string.

natural frequencies: $\quad \omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$ //

mode shapes: $\quad U_n = A \sin \frac{n\pi x}{L}$ //

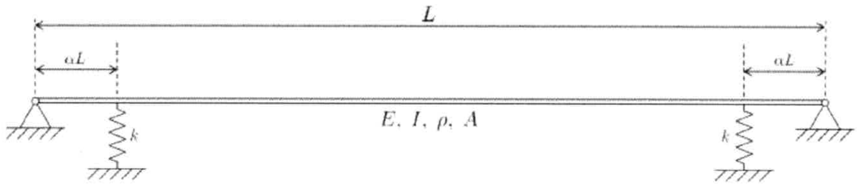
(ii) What would be the effect on the mode shapes and natural frequencies if the thickness of the whole beam was increased? Justify your answer. [10%]

no change to mode shapes, as not function of d //

$\omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \quad ; \quad I \propto d^3$

$A \propto d$

so $\quad \omega_n \propto d \quad$ so increase in thickness $\Rightarrow$ increase in ω_n //



(b) It is necessary to modify the bridge so that the first natural frequency is higher. It is proposed that two springs are added to the structure as illustrated in Figure 2. The springs are both of stiffness k and they are placed a distance αL from each end.

(i) Assuming sinusoidal mode shapes, use Rayleigh's principle to estimate the factor by which the natural frequencies of the beam are increased. [40%]

$$\text{Rayleigh quotient} = \frac{V}{\tilde{T}} \simeq \omega_n^2$$ (if mode shape good approximation)

let $$y(x) = \sin\frac{n\pi x}{L}$$ (in fact these are the original mode shapes of the unmodified beam)

$$y''(x) = -\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}$$

$$\left(y''(x)\right)^2 = \left(\frac{n\pi}{L}\right)^4 \left(\sin\frac{n\pi x}{L}\right)^2$$

hence $$V = \frac{1}{2}EI\left(\frac{n\pi}{L}\right)^4 \int_0^L \left(\sin\frac{n\pi x}{L}\right)^2 dx + 2.\frac{1}{2}k\left(\sin n\pi\alpha\right)^2.$$

$$= \frac{1}{4}EI\left(\frac{n\pi}{L}\right)^4 \int_0^L \left(1 - \underbrace{\cos\frac{2n\pi x}{L}}_{\text{integral over whole period} = 0.}\right) dx + k\left(\sin n\pi\alpha\right)^2.$$

$$= \underbrace{\frac{1}{4}EI\left(\frac{n\pi}{L}\right)^4 L}_{V_0} + \underbrace{k\left(\sin n\pi\alpha\right)^2}_{\Delta V}.$$

$\tilde{T}$ unchanged. , $V = V_0 + \Delta V$, $\omega_{new}^2 = \dfrac{V_0 + \Delta V}{\tilde{T}}$

$$\omega_{new}^2 = \frac{V_0 \left(1 + \Delta V / V_0\right)}{\tilde{T}}$$

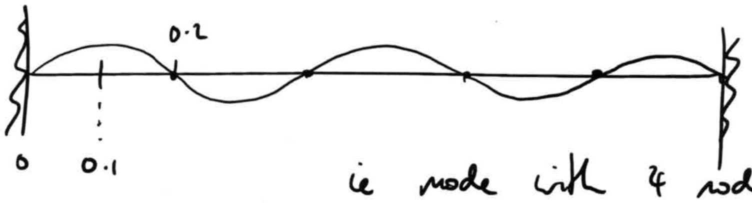
$$\omega_{new}^2 = \omega_n^2 \underbrace{\left(1 + \Delta V / V_0\right)}_{\text{factor by which square of natural frequencies increased}} = \omega_n^2 f^2$$

$$f^2 = 1 + \frac{\Delta V}{V_0} = 1 + \frac{k\left(\sin n\pi\alpha\right)^2}{\frac{1}{4} EI \left(\frac{n\pi}{L}\right)^4 L} = 1 + \frac{4kL^3 \left(\sin n\pi\alpha\right)^2}{EI\,(n\pi)^4}$$

(ii) For $\alpha = 0.1$, which natural frequencies would be most affected in terms of their absolute frequency change (i.e. not in terms of their fractional change)? [10%]

modes with antinode @ $x = \alpha L = 0.1L$

the first example is:



ie mode with 4 nodes, $n = 5$.

more generally: $\Delta V = k\left(\sin n\pi\alpha\right)^2$

so when $\left|\sin n\pi\alpha\right| = 1$

$$0.1 n\pi = \pi/2 + n\pi$$

$$n\pi = 5\pi + 10n\pi.$$

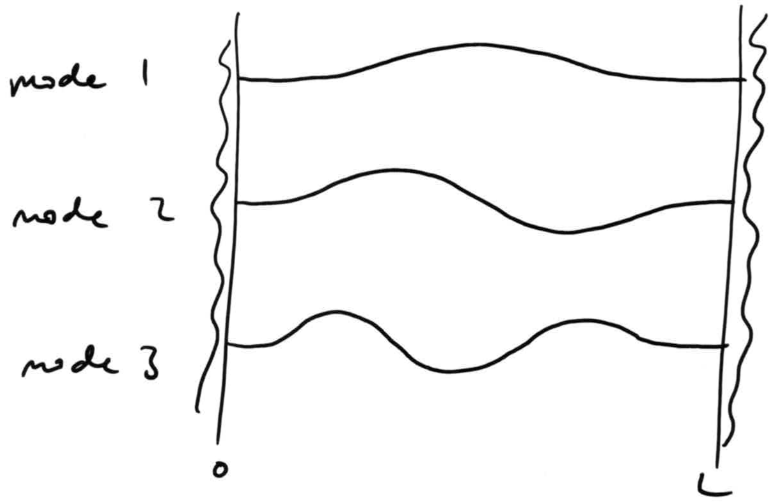
$$n = 5 + 10n.$$

$$n = 5, 15, 25, \ldots$$

[If considering fractional change then first mode most affected because of $1/n^4$ factor.]

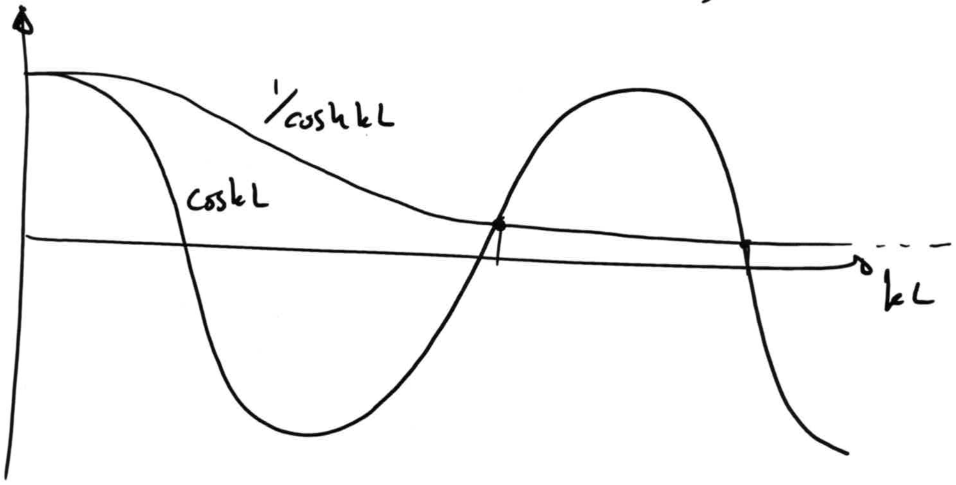
(iii) Without detailed calculation, what would happen to the natural frequencies and mode shapes in the limit as $k \to \infty$ with α small but not zero. [20%]

Boundary conditions effectively become clamps, so mode shapes:



Natural frequencies tend to clamped-clamped frequencies, ie solutions to $\cos kL . \cosh kL = 1$, $\omega = k^2\sqrt{\frac{EI}{\rho A}}$

for interest

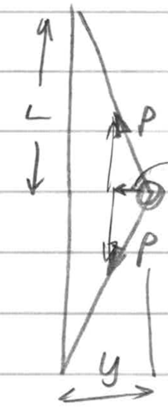


$$k_n \simeq \frac{\pi}{2} + n\pi \quad \text{for } n \geq 1$$

$$= (n + 1/2)\pi$$

$$\omega_n \simeq \left[(n+1/2)\pi\right]^2 \sqrt{EI/\rho A}$$

3 (a) The tension in the wire generates a side force which acts like a lateral spring joining the masses.



$2P\theta = 2P\dfrac{y}{L} = \left(\dfrac{2P}{L}\right)y$ So the stiffness of each side of the wire is P/L.

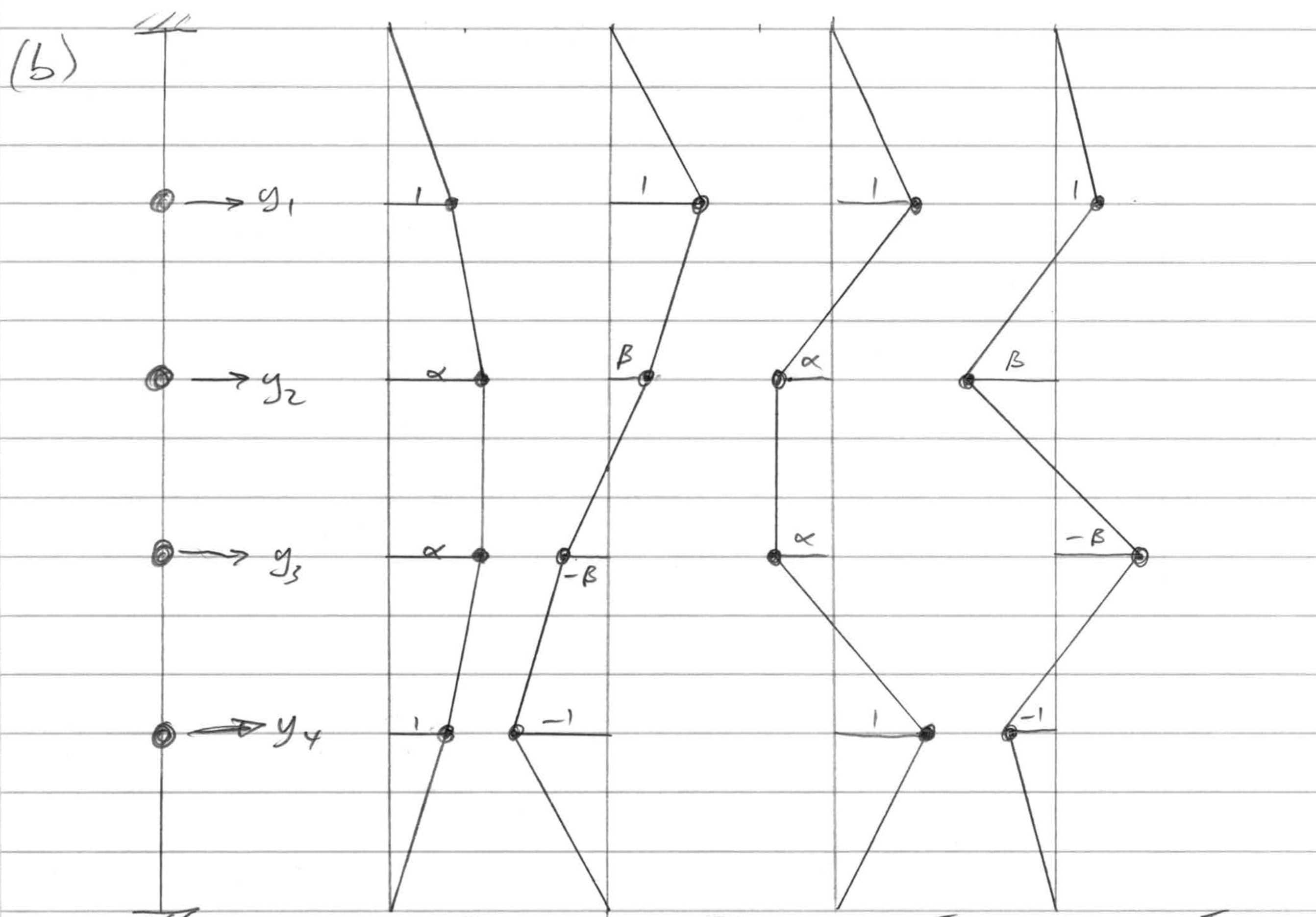
Therefore the potential energy for lateral vibration is

$$V = \frac{1}{2}\frac{P}{L}\left[y_1^2 + (y_2-y_1)^2 + (y_3-y_2)^2 + (y_4-y_3)^2 + y_4^2\right]$$

$$= \frac{P}{L}\left[y_1^2 + y_2^2 + y_3^2 + y_4^2 - y_1y_2 - y_2y_3 - y_3y_4\right]$$

$$T = \frac{1}{2}m\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 + \dot{y}_4^2\right)$$

(b)



$[1\ \alpha\ \alpha\ 1]^T$ $[1\ \beta\ -\beta\ -1]^T$ $[1\ \alpha\ \alpha\ 1]^T$ $[1\ \beta\ -\beta\ -1]^T$

1 2 3 4

Modes 1 and 3 are both of the form $[1\ \alpha\ \alpha\ 1]^T$

3 Cont.

(c) Rayleigh $\omega^2 = \dfrac{V_{max}}{T^*}$

Use the mode shape $[1\ \alpha\ \alpha\ 1]^T$:

$$\omega^2 = \frac{\frac{1}{2}P/L\left(1^2 + (\alpha-1)^2 + \cancel{(\alpha-\alpha)^2} + (1-\alpha)^2 + 1^2\right)}{\frac{1}{2}m\left(1^2 + \alpha^2 + \alpha^2 + 1^2\right)}$$

$$= \frac{P}{Lm}\frac{\left(\cancel{2} + \cancel{2}(\alpha-1)^2\right)}{\left(\cancel{2} + \cancel{2}\alpha^2\right)} = \frac{P}{Lm}\frac{\left(1+(\alpha-1)^2\right)}{1+\alpha^2}$$

to find α differentiate : $d\omega^2/d\alpha = 0$ (gives the minimum ω^2)

$$\Rightarrow \frac{(1+\alpha^2)\cancel{2}(\alpha-1) - (1+(\alpha-1)^2)\cancel{2}\alpha}{\cancel{(1+\alpha^2)^2}} = 0$$

$$(1+\alpha^2)(\alpha-1) - (1+(\alpha-1)^2)\alpha = 0$$

$$\cancel{\alpha^3} - \alpha^2 + \alpha - 1 - (\alpha + \cancel{\alpha^3} - 2\alpha^2 + \alpha) = 0$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \begin{array}{l} -0.618 \\ +1.618 \end{array}$$

So the first and third modes are

$$\underset{\sim}{u}^{(1)} = [1\quad 1.618\quad 1.618\quad 1]^T \qquad \alpha = 1.618$$

$$\underset{\sim}{u}^{(3)} = [1\quad -0.618\quad -0.618\quad 1]^T \qquad \alpha = -0.618$$

and the natural frequencies are

$\alpha = 1.618$ $$\omega_1^2 = \frac{P}{Lm}\frac{\left(1 + 0.618^2\right)}{1+1.618^2} = 0.382\frac{P}{Lm} \qquad \omega_1 = 0.618\sqrt{\frac{P}{Lm}}$$

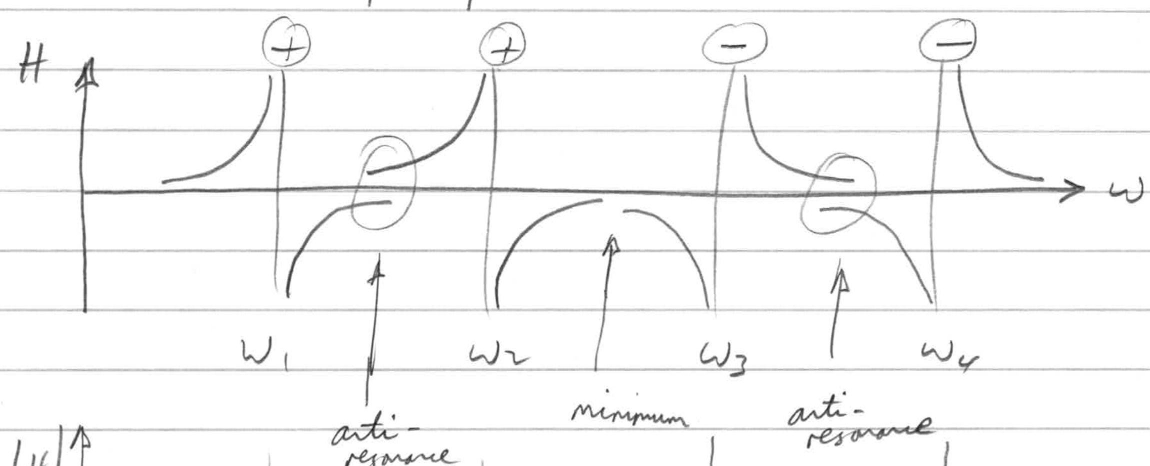
$\alpha = -0.618$ $$\omega_3^2 = \frac{P}{Lm}\frac{\left(1 + (-1.618)^2\right)}{1+(0.618)^2} = 2.618\frac{P}{Lm} \qquad \omega_3 = 1.618\sqrt{\frac{P}{Lm}}$$

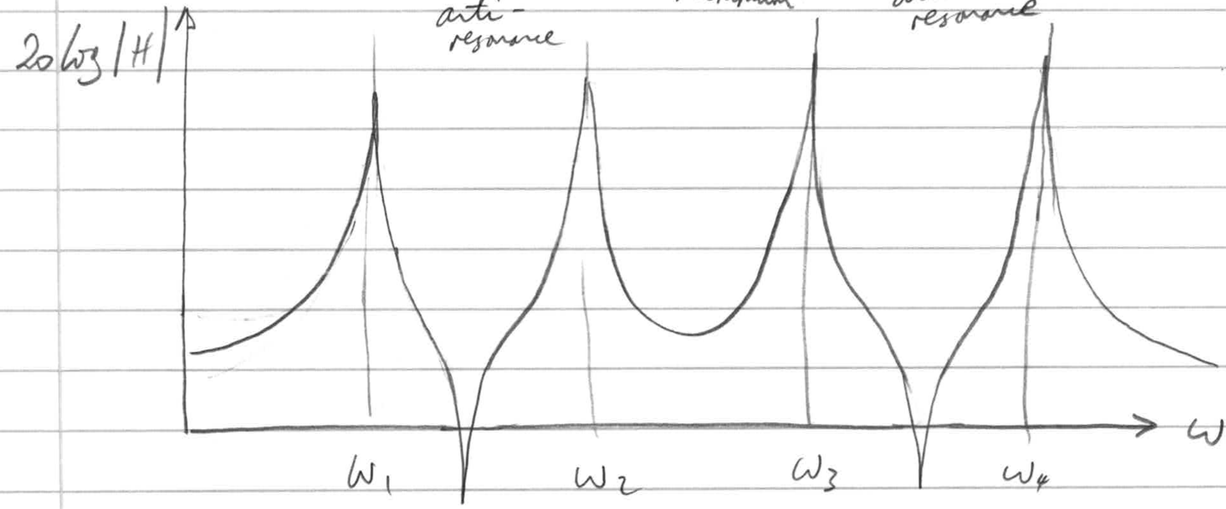
Transfer function

$$H_{21} = \frac{Y_2}{X_1} = \sum_{n=1}^{N} \frac{u_2^{(n)} u_1^{(n)}}{\omega_n^2 - \omega^2}$$

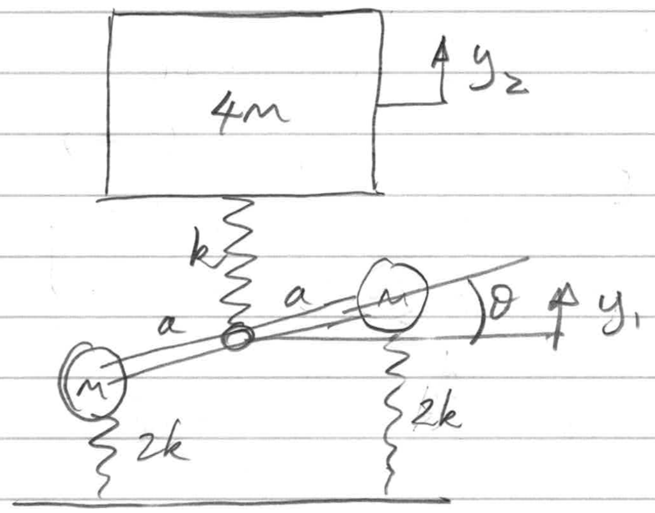
For masses 1 & 2, the modal products have signs

$+ \quad + \quad - \quad -$





4.



(a) $V = \frac{1}{2}k(y_2 - y_1)^2 + \frac{1}{2}2k(y_1 + a\theta)^2 + \frac{1}{2}2k(y_1 - a\theta)^2$

$= \frac{1}{2}k\left[y_2^2 + y_1^2 - 2y_1y_2 + 2y_1^2 + \cancel{4y_1a\theta} + 2a^2\theta^2 + 2y_1^2 - \cancel{4y_1a\theta} + 2a^2\theta^2\right]$

$= \frac{1}{2}k\left[y_2^2 + 5y_1^2 + 4a^2\theta^2 - 2y_1y_2\right]$

$= \frac{1}{2}\begin{bmatrix} y_1 & y_2 & \theta\end{bmatrix}\begin{bmatrix} 5k & -k & 0 \\ -k & k & 0 \\ 0 & 0 & 4a^2k\end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ \theta\end{Bmatrix}$

$T = \frac{1}{2}2m\dot{y}_1^2 + \frac{1}{2}4m\dot{y}_2^2 + \frac{1}{2}I\dot{\theta}^2$

with $I = 2a^2m$

$= \frac{1}{2}\begin{bmatrix} \dot{y}_1 & \dot{y}_2 & \dot{\theta}\end{bmatrix}\begin{bmatrix} 2m & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & 2a^2m\end{bmatrix}\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}\end{Bmatrix}$

(b) <u>Natural motion</u> $([K] - \omega^2[M])\underline{u} = 0$

$$\begin{vmatrix} 5k - \omega^2 2m & -k & 0 \\ -k & k - \omega^2 4m & 0 \\ 0 & 0 & 4a^2k - \omega^2 2a^2m \end{vmatrix} = 0$$

$\Rightarrow (5k - \omega^2 2m)\left[(k - \omega^2 4m)(4a^2k - \omega^2 2a^2m) - k^2(4a^2k - \omega^2 2a^2m)\right] = 0$

4(b) Cont

$$(4a^2k - \omega^2 2a^2m)\left[(5k - \omega^2 2m)(k - \omega^2 4m) - k^2\right] = 0$$

$\Rightarrow$ One root is $\omega^2 = \dfrac{4a^2k}{2a^2m} = \dfrac{2k}{m}$

The other two roots are solutions of

$$(5k^2 - 22\omega^2 mk + 8m^2\omega^4 - k^2) = 0.$$

$$8m^2\omega^4 - 22mk\,\omega^2 + 4k^2 = 0$$

$$\omega^2 = \frac{22mk \pm \sqrt{22^2m^2k^2 - 128m^2k^2}}{16m^2}$$

$$= \left[\frac{11 \pm \sqrt{89}}{8}\right]\frac{k}{m} \qquad = \begin{Bmatrix} 0.196 \\ 2.554 \end{Bmatrix}\frac{k}{m}$$

<u>Mode Shapes</u>

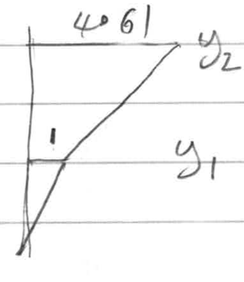
The uncoupled mode with $\omega_2^2 = 2k/m$ has $\underline{u}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$
(pure pitch of walking beam)

The other two modes are solutions of

$$\begin{bmatrix} 5k - \omega^2 2m & -k \\ -k & k - \omega^2 4m \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \underline{0}$$

First row: $(5k - 2\omega^2 m)y_1 - ky_2 = 0$

So the e-vector is $\dfrac{y_2}{y_1} = \dfrac{5k - 2\omega^2 m}{k}$

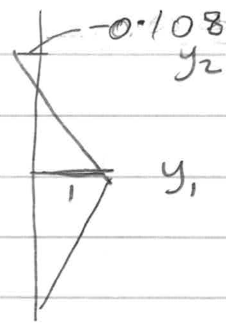


for $\omega_1^2 = 0.196k/m$, $\dfrac{y_2}{y_1} = \dfrac{5 - 2(0.196)}{1} = 4.610$ (body bounce)

4. Cont

for $\omega_3^2 = 2.554\frac{k}{m}$, $\frac{y_2}{y_1} = \frac{5-2(2.554)}{1} = -0.108$

(wheel hop)



(c) Added mass $4m \rightarrow 4m(1+\varepsilon)$

Rayleigh: $\omega^2 \approx \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k(y_2^2+5y_1^2+4a^2\theta^2-2y_1y_2)}{\frac{1}{2}m(2y_1^2+4(1+\varepsilon)y_2+2a^2\theta^2)}$

Use the same mode shapes to find small changes in freq.

Body bounce mode $y_1=1$, $y_2=4.610$, $\theta=0$

$\Rightarrow \omega_1^2 \rightarrow \frac{k}{m}\frac{(4.61^2+(5)1^2-2(4.61)(1))}{2\times1^2+4(1+\varepsilon)(4.61)^2} = \frac{k}{m}\frac{17.032}{87.0+85\varepsilon}$

$\varepsilon=0$ $\quad \omega_1^2 = \frac{17.032}{87}\frac{k}{m} = 0.196\,k/m$ ✓ $\quad \omega_1 = 0.443\sqrt{\frac{k}{m}}$

To reduce ω_1 by 10%, (to $0.9\times0.443\sqrt{\frac{k}{m}}$)

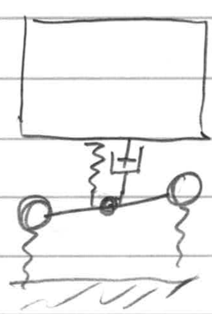
$$\omega_1^2 = 0.196\times0.9^2\frac{k}{m} = \frac{17.032}{87+85\varepsilon}$$

$$\Rightarrow 87+85\varepsilon = \frac{17.032}{(0.196)(0.9)^2}$$

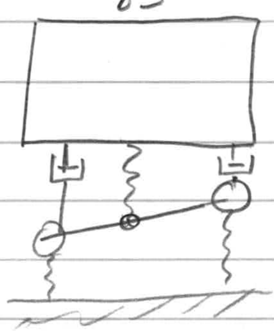
$$\varepsilon = \frac{\frac{17.032}{(0.196)(0.9)^2}-87}{85} = 0.24 \text{ ie } \underline{\underline{24\%}}$$

(New spring mass is $4(1.24)m = 4.96m$)

(d)



Damps modes 1 & 3 but not bogie pitch mode



Damps all modes

ENGINEERING TRIPOS PART IIA – 2017   Module 3C6 – Vibration

Assessor's Comments – D. Cebon 19/5/2017

Question 1 – Coupled strings

Not attempted by many candidates, but those who tackled this one generally did well. Most found mode shapes and natural frequencies for the string. When deriving the transfer function in (a)(iii) many forgot that the mode shapes needed to be mass normalized. Most applied the coupling formula correctly for (b)(i) and saw that mode 3 would not appear in the sketch of the transfer function. In (b)(iii) most candidates drew the standard sequence of modes for a single string, only a few saw that the extra modes would be those with nodes at the coupling point and would be antisymmetric for the pair of strings.

Question 2 – Beam with added springs

Attempted by most candidates, generally good solutions. Most derived the mode shapes and frequencies for a beam, but not all saw how frequency scaled with beam thickness. Application of Rayleigh's principle was highly variable: most knew the basic formulation but did not apply it accurately. Note that the examiner allowed for several definitions of 'factor increase' as long as the candidates made clear their definition. Most candidates could see that the most affected modes would be when the spring was at an antinode of a mode, but when listing frequencies included modes with a node at the spring. For (b)(iii) only a few saw that the frequencies and mode shapes would tend to that of a clamped beam, rather than a pinned beam of shorter length.

Question 3 – 4 beads on a string

Attempted by almost all candidates and mostly well done.

Question 4 – Natural modes of a 3 DoF system

Attempted by most candidates and generally well done. Part (b) was reasonably competent, with most candidates finding the correct eigenvalues. Quite a few couldn't find the correct eigenvectors. Rayleigh was generally good. The proposed design changes showed reasonable insight, though quite a few candidates thought that putting dampers in parallel with the tyres would be a good plan…