3C7 - 2015

1) (a) $$\phi = \frac{\sigma_0}{\alpha^2}\cos(\alpha x)(1+\alpha y)e^{-\alpha y} \quad , \alpha = \frac{2\pi}{\lambda}$$

$$\frac{\partial^2\phi}{\partial x^2} = -\sigma_0 \cos(\alpha x)\left((1+\alpha y)e^{-\alpha y}\right)$$

$$\frac{\partial\phi}{\partial y} = \frac{\sigma_0}{\alpha^2}\cos(\alpha x)\left[\alpha e^{-\alpha y} - \alpha(1+\alpha y)e^{-\alpha y}\right]$$

$$\cancel{\frac{\partial^2\phi}{\partial y^2} = \frac{\sigma_0}{\alpha^2}}$$

$$= -\sigma_0 \cos(\alpha x)\left[y e^{-\alpha y}\right]$$

$$\frac{\partial^2\phi}{\partial y^2} = -\sigma_0\cos(\alpha x)\left[(1-\alpha y)e^{-\alpha y}\right]$$

$$\frac{\partial^2\phi}{\partial x\partial y} = \sigma_0 \sin(\alpha x)\left[\alpha y\, e^{-\alpha y}\right]$$

$$\therefore \quad \sigma_{xx} = -\sigma_0\cos(\alpha x)\left[(1-\alpha y)e^{-\alpha y}\right]$$

$$\sigma_{yy} = -\sigma_0\cos(\alpha x)\left[(1+\alpha y)e^{-\alpha y}\right]$$

$$\sigma_{xy} = -\sigma_0\sin(\alpha x)\left[\alpha y\, e^{-\alpha y}\right]$$

Check for compatibility

$$f: \sigma_{xx} + \sigma_{yy} = -2\sigma_0 \cos(\alpha x)\, e^{-\alpha y}$$

$$\frac{\partial^2 f}{\partial x^2} = 2\sigma_0 \alpha^2 \cos(\alpha x)\, e^{-\alpha y}$$

$$\frac{\partial^2 f}{\partial y^2} = -2\sigma_0 \alpha^2 \cos \alpha x \, e^{-\alpha y}$$

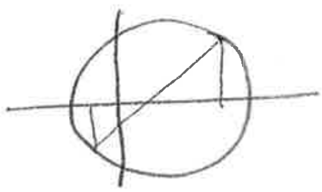
$$\Rightarrow \quad \nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \quad ;$$

Check boundary conditions

$\sigma_{xy} = 0$ on $y = 0$ ✓

σ_{yy} on $y = 0$ $\Rightarrow$ $-\sigma_0 \cos(\alpha x)$ ✓

(b) Diameter of Mohr's circle



$$D^2 = (\sigma_{xx} - \sigma_{yy})^2 + (2\sigma_{xy})^2$$

$$= \left[2\sigma_0 \cos(\alpha x)\, \alpha y\, e^{-\alpha y}\right]^2 + \left[2\sigma_0 \sin(\alpha)\, y\, e^{-\alpha y}\right]^2$$

$$= 4\sigma_0^2 e^{-2\alpha y} (\alpha y)^2$$

$$D = 2\sigma_0 \alpha y\, e^{-\alpha y}$$ independent of x

$$k = \frac{D}{2}$$

$$k = \sigma_0 (\alpha y) e^{-\alpha y}$$

Max at $y = \frac{1}{\alpha}$

$$\Rightarrow \sigma_0 = k e$$

(c) Mohr's circle biggest at $\alpha y = 1$

von Mises criterion is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6k^2$$

$\sigma_3 = 0 \because$ plane stress

$$(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 = 6k^2$$

$$\Rightarrow D^2 + \left(\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{D}{2}\right)^2 + \left(\frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{D}{2}\right)^2 = 6k^2$$

$$\Rightarrow D^2 + 2\left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + 2\left(\frac{D}{2}\right)^2 = 6k^2$$

D is max at $\alpha y = 1$

$\sigma_{yy} + \sigma_{xx}$ max @ $x = 0 \Rightarrow \sigma_{xy} = 0$

$\Rightarrow \sigma_3 = \sigma_2 = 0 \quad \sigma_1 = \sigma_{yy} = -\frac{2\sigma_0}{e}$

$$\frac{4\sigma_0^2}{e^2} = 6k^2$$

$$\sigma_0 = \sqrt{\frac{3}{4}}\, e k$$

$$\sigma_0 = \frac{\sqrt{3}}{2}\, k e$$

2(a)

$$\frac{\partial \phi}{\partial r} = \frac{2Mr}{a^2} + \frac{N}{r}$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{2M}{a^2} - \frac{N}{r^2}$$

$$\sigma_{rr} = \frac{2M}{a^2} + \frac{N}{r^2}$$

$$\sigma_{\theta\theta} = \frac{2M}{a^2} - \frac{N}{r^2}$$

$$\sigma_{r\theta} = 0$$

Matches Lames with choice $A = \frac{2M}{a^2}$, $B = -N$

(b) $r \to \infty$ $\quad \sigma_{rr} = \sigma_0 = \sigma_{\theta\theta}$

assume hole of radius a

at $r = a$, $\sigma_{rr} = 0$

$\sigma_{rr} = A - \frac{B}{r^2}$, $\quad \sigma_{\theta\theta} = A + \frac{B}{r^2}$

$A = \sigma_0$, $\quad B = \sigma_0 a^2$

$$\sigma_{rr} = \sigma_0 \left(1 - \left(\frac{a}{r}\right)^2\right)$$

$$\sigma_{\theta\theta} = \sigma_0 \left(1 + \left(\frac{a}{r}\right)^2\right)$$

SCF $= 2$

(c)

$$a\frac{\partial\phi}{\partial r} = a^2\sigma_0\cos 2\theta\left(-\left(\frac{r}{a}\right)+\left(\frac{a}{r}\right)^3\right)$$

$$a^2\frac{\partial^2\phi}{\partial r^2} = -a^2\sigma_0\cos 2\theta\left[1+\frac{3a^4}{r^4}\right]$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = a\sigma_0\sin 2\theta\left[\left(\frac{r}{a}\right)-\frac{2a}{r}+\left(\frac{a^3}{r^3}\right)\right]$$

$$\frac{\partial^2\phi}{\partial\theta^2} = a^2\sigma_0\cos 2\theta\left[2\left(\frac{r}{a}\right)^2-4+2\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_{rr} = \sigma_0\cos 2\theta\left[-1+\left(\frac{a}{r}\right)^4-4\left(\frac{a}{r}\right)^2+2+2\left(\frac{a}{r}\right)^4\right]$$

$$= \sigma_0\cos 2\theta\left[1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]$$

$r=a$, $\sigma_{rr}=0$ ✓

$r\to\infty$, $\sigma_{rr}=\sigma_0\cos 2\theta$

$$\sigma_{\theta\theta} = -\sigma_0\cos 2\theta\left(1+3\left(\frac{a}{r}\right)^4\right]$$

$r\to\infty$ $\quad\sigma_{\theta\theta} = -\sigma_0\cos 2\theta$

$$\sigma_{r\theta} = \sigma_0\sin 2\theta\left[-1-2\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]$$

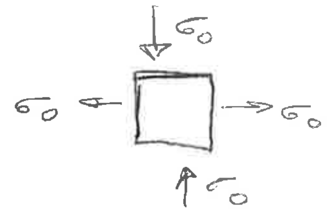
$r = a \quad \sigma_{r\theta} = 0 \quad \Rightarrow$ satisfies B C on hole

Remotely

$$\sigma_{rr} = \sigma_0 \cos 2\theta$$

$$\sigma_{\theta\theta} = -\sigma_0 \cos 2\theta$$

$$\sigma_{r\theta} = -\sigma_0 \sin 2\theta$$



$\Rightarrow$ pure shear

at edge of hole for $\theta = \frac{\pi}{2}$

$$\sigma_{\theta\theta} = 4\sigma_0$$

$$\Rightarrow SCF = 4.$$

3(a) $$u = Cr + \frac{D}{r}$$

$$\varepsilon_{rr} = \frac{du}{dr} = C - \frac{D}{r^2}$$

$$\epsilon_{\theta\theta} = \frac{u}{r} = C + \frac{D}{r^2}$$

equilibrium is $\sigma_{\theta\theta} = \frac{d}{dr}\left(r\sigma_{rr}\right)$

$$\sigma_{rr} = \frac{\overline{E}}{1-\bar{\nu}^2}\left[\varepsilon_{rr} + \bar{\nu}\,\varepsilon_{\theta\theta}\right]$$

$$= \frac{\overline{E}}{1-\bar{\nu}^2}\left[C - \frac{D}{r^2} + \bar{\nu}\left(C + \frac{D}{r^2}\right)\right]$$

$$= \frac{\overline{E}}{1-\bar{\nu}^2}\left[(1+\bar{\nu})C - \frac{D}{r^2}(1-\bar{\nu})\right] = C' - \frac{D'}{r^2}$$

$$\sigma_{\theta\theta} = \frac{\overline{E}}{1-\bar{\nu}^2}\left[C + \frac{D}{r^2} + \bar{\nu}\left(C - \frac{D}{r^2}\right)\right]$$

$$= \frac{\overline{E}}{1-\bar{\nu}^2}\left[C(1+\bar{\nu}) + \frac{D}{r^2}(1-\bar{\nu})\right] = C' + \frac{D'}{r^2}$$

$$\frac{d}{dr}\left(rC' - \frac{D'}{r}\right) = C' + \frac{D'}{r^2} = \sigma_{\theta\theta} \quad ✓$$

ie satisfies equilibrium.

b (i) outer radius unconstrained $\Rightarrow \sigma_{rr}\big|_{r=2a} = 0$

$$C' = \frac{D'}{4a^2}$$

$$D' = 4a^2 C'$$

$$(1-\bar{\nu})\frac{D\cancel{\bar{E}}}{\cancel{1-\bar{\nu}^2}} = 4a^2 \frac{\cancel{\bar{E}}(1+\bar{\nu})C}{\cancel{1-\bar{\nu}^2}}$$

$$D = 4a^2 \frac{(1+\bar{\nu})}{(1-\bar{\nu})}$$

$$1+\bar{\nu} = 1 + \frac{\nu}{1-\nu} = \frac{1}{1-\nu}$$

$$1-\bar{\nu} = 1 - \frac{\nu}{1-\nu} = \frac{1-2\nu}{1-\nu}$$

$$D = \frac{4a^2}{1-2\nu} C$$

$$\Rightarrow u = Cr + \frac{4a^2 C}{1-2\nu}\frac{1}{r}$$

at $r = a$, $u = 0.01a$

$$0.01a = C\left[\frac{a(1-2\nu) + 4a}{1-2\nu}\right]$$

$$C = 0.01\left(\frac{1-2\nu}{5-2\nu}\right)$$

$$D = 4a^2 \frac{0.01}{5-2\nu}$$

at $r = 2a$

$$u = \frac{0.01}{5-2\nu}\left[(1-2\nu)2a + \frac{4a^2}{2a}\right]$$

$$= 0.04a\left(\frac{1-\nu}{5-2\nu}\right) + \frac{0.02a}{(5-2\nu)}$$

(ii) $\quad u(2a) = 0$

$$0 = C\,2a + \frac{D}{2a} \quad \Rightarrow \quad D = -4a^2 C$$

$r = a, \quad u = 0.01a$

$$0.01a = Ca - \frac{4a^2C}{a} = -3Ca$$

$$C = -\frac{0.01}{3}, \quad D = \frac{0.04a^2}{3}$$

$$\sigma_{rr}\Big|_{r=2a} = \bar{E}\left[\frac{-\frac{0.01}{3}}{1-\bar{\nu}} - \frac{\frac{0.01}{3}}{1+\bar{\nu}}\right]$$

$$= \frac{0.01\bar{E}}{3}\left[-\frac{1}{1-\bar{\nu}} \cancel{*} - \frac{1}{1+\bar{\nu}}\right]$$

$(1-\nu)(1-2\nu)$

$1-\nu-2\nu+2\nu^2$

$1-3\nu+2\nu^2$

$$= -\frac{0.01}{3}\frac{E}{(1-\nu^2)}\left[\frac{1-\nu}{1-2\nu} + 1-\nu\right]$$

$$= -\frac{0.01}{3}\frac{E}{(1+\nu)(1-\nu)}\left[\frac{2-4\nu+2\nu^2}{1-2\nu}\right]$$

$$= -0.01\times\frac{2}{3}\,E\left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)$$

4(a)

$$\dot{\varepsilon}_{rr} = \frac{du}{dr} = -\frac{A}{r^2}$$

$$\dot{\varepsilon}_{\theta\theta} = \frac{u}{r} = \frac{A}{r^2}$$

$$\dot{\varepsilon}_{zz} = 0$$

$$\dot{w} = \sigma_{\theta\theta}\dot{\varepsilon}_{\theta\theta} + \sigma_{rr}\dot{\varepsilon}_{rr}$$

$$= (\sigma_{\theta\theta} - \sigma_{rr})\frac{A}{r^2}$$

But since $\dot{\varepsilon}_{rr} = -\dot{\varepsilon}_{\theta\theta}$ => active yield criterion is

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_T$$

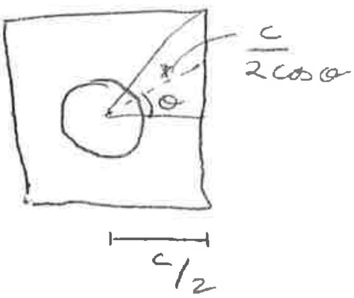
$$\Rightarrow \quad \dot{W} = \int_a^b \frac{\sigma_T A}{r^2} 2\pi r\,dr = 2\pi A \sigma_T \ln\left(\frac{b}{a}\right)$$

work done by pressure $= 2\pi a\, p\, u|_{r=a} = 2\pi a\, p \frac{A}{a}$

$$\Rightarrow \quad 2\pi p A = 2\pi A \sigma_Y \ln\left(\frac{b}{a}\right)$$

$$p = \sigma_Y \ln\left(\frac{b}{a}\right)$$

(b)



$$\dot{W} = 8\int_0^{\pi/4}\int_a^{\frac{c}{2\cos\theta}} \frac{\sigma_Y A}{r^2}\, r\,dr\,d\theta$$

$$= 8\sigma_Y A \int_0^{\pi/4} \ln\left(\frac{c}{2a\cos\theta}\right) d\theta$$

$$= 8\sigma_Y A \left[\frac{\pi}{4}\ln\left(\frac{c}{2a}\right) - \int_0^{\pi/4} \ln(\cos\theta)\, d\theta\right]$$

$$\dot{W} = 8\sigma_Y A \left[\frac{\pi}{4}\ln\left(\frac{c}{2a}\right) + 0.086\right]$$

$$p\, 2\pi \cancel{A} = 8\sigma_Y \cancel{A}\left[\frac{\pi}{4}\ln\left(\frac{c}{2a}\right) + 0.086\right]$$

$$p = \frac{4\sigma_Y}{\pi}\left[\frac{\pi}{4}\ln\left(\frac{c}{2a}\right) + 0.086\right]$$

(c) The solution is ~~exact~~ expected to be exact for the circular plate as the velocity field used is axisymmetric ~~that~~ & problem too is ~~ax~~ axi-symmetric but approximate (ie upper bound) for the square plate.

Comments on questions

Q1 Airy stress function in a half-space & application of yield criteria

A question that was reasonably well-attempted. Most students struggled on part (c) in attempting to use the von-Mises criterion to predict first yield. They were unable to determine the location at which yielding initiated in this half-space.

Q2 Airy stress functions in polar co-ordinates

Generally the students had no problems with this question with a number of candidates being awarded 100% on this question. Some students struggled in the co-ordinate rotation to determine the SCF under pure shear.

Q3 Axi-symmetric elastic fields

While most candidates answered part (a) well they struggled with both subsections of part (b). They struggled to correctly apply the boundary conditions as specified in the question.

Q4 Upper bound plastic collapse calculation

A very unpopular question. The assessor is surprised at this given that part (a) was basically covered in the lecture notes and was a direct application of the upper bound method. Of the 6 students who attempted it 2 were awarded 100% on the question.