3C7 - 2016

1)

(a) $$\sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha}{r^2}\int_0^r r T_0\left(1-\frac{r^2}{b^2}\right)dr$$ - for plane stress

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} + \frac{E\alpha}{r^2}\int_0^r r T_0\left(1-\frac{r^2}{b^2}\right)dr - E\alpha T_0\left(1-\frac{r^2}{b^2}\right)$$

$$\int_0^r r T_0\left(1-\frac{r^2}{b^2}\right)dr = \frac{1}{2}T_0 r^2\left(1-\frac{1}{2}\frac{r^2}{b^2}\right)$$

B.C. σ_{rr} finite at $r=0 \Rightarrow B=0$

$\sigma_{rr}=0$ @ $r=b \Rightarrow A = \frac{E\alpha T_0}{4}$

$$\Rightarrow \sigma_{rr} = -\frac{E\alpha T_0}{4}\left(1-\frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = \frac{E\alpha T_0}{4} + \frac{E\alpha T_0}{2}\left(1-\frac{1}{2}\frac{r^2}{b^2}\right) - E\alpha T_0\left(1-\frac{r^2}{b^2}\right)$$

$$= -\frac{E\alpha T_0}{4}\left(1-\frac{3r^2}{b^2}\right)$$

Now convert to plane strain

$$\sigma_{rr} = -\frac{E\alpha T_0}{4(1-\nu)}\left(1-\frac{r^2}{b^2}\right) \; ; \quad \sigma_{\theta\theta} = -\frac{E\alpha T_0}{4(1-\nu)}\left(1-\frac{3r^2}{b^2}\right)$$

$\varepsilon_{zz}=0 \Rightarrow \sigma_{zz} = \nu(\sigma_{\theta\theta}+\sigma_{rr}) = -\frac{\nu E\alpha T_0}{2(1-\nu)}\left(1-2\frac{r^2}{b^2}\right)$

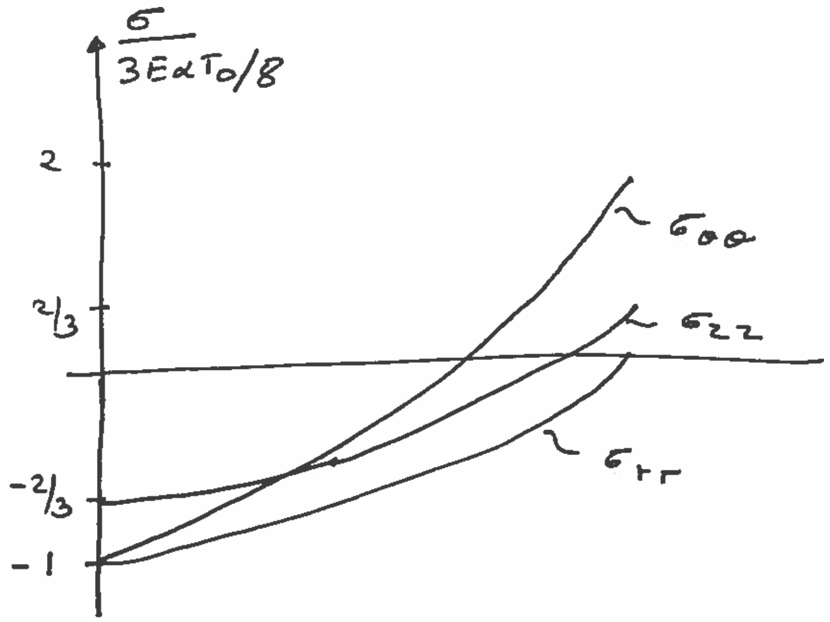
$\sigma_{r\theta}=0$

(b) $\nu = \frac{1}{3}$

$$\sigma_{rr} = -\frac{3E\alpha T_0}{8}\left(1-\frac{r^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = -\frac{3\alpha E T_0}{8}\left(1-3\frac{r^2}{b^2}\right)$$

$$\sigma_{zz} = -\frac{E\alpha T_0}{4}\left(1-2\frac{r^2}{b^2}\right)$$



(c) $$\frac{u}{r} = \varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr} - \nu\sigma_{zz}\right)$$

$$u = \frac{r}{E}\left(\frac{-E\alpha T_0}{4(1-\nu)}\right)\left[1-\frac{3r^2}{b^2} - \nu\left(1-\frac{r^2}{b^2}\right) - 2\nu^2\left(1-\frac{2r^2}{b^2}\right)\right]$$

@ $r=b$ $$u = \frac{-\alpha b T_0}{4(1-\nu)}\left(-2+2\nu^2\right) = \frac{\alpha b T_0}{2}(1+\nu)$$

(d)

$$\tau_{max} = \max\left[\frac{|\sigma_{rr} - \sigma_{\theta\theta}|}{2}, \frac{|\sigma_{\theta\theta} - \sigma_{zz}|}{2}, \frac{|\sigma_{zz} - \sigma_{rr}|}{2} \right]$$

Inspection shows (see b) τ_{max} occurs at $r = b$ with

$$\tau_{max} = \frac{Y}{2} = \frac{|\sigma_{rr} - \sigma_{\theta\theta}|}{2} = \frac{E\alpha T_0}{4(1-\nu)}$$

If no yielding max allowable T_0 is

$$T_0 = \frac{2(1-\nu)Y}{E\alpha}$$

(2)

(a) ① $\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + F_x = 0$ &

② $\dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + F_y = 0$ where $F_x = 0;\ F_y = -\rho g$

① is automatically satisfied ;

② $\rho g - \rho g = 0$ ✓   at $y = 0$ $-\sigma_{yy} A = \rho g h A$

$\Rightarrow C = -\rho g h$

$$\sigma_{yy} = \rho g\,(y - h)$$

(b) $\varepsilon_{xx} = -\dfrac{\nu \sigma_{yy}}{E} = \dfrac{\partial u}{\partial x} = -\dfrac{\nu \rho g}{E}(y-h)$

$$u = \frac{\nu \rho g}{E}(h-y)x + f(y)$$

$\varepsilon_{yy} = \dfrac{\sigma_{yy}}{E} = \dfrac{\partial v}{\partial y} = \dfrac{\rho g}{E}(y-h)$

$$\Rightarrow v = \frac{\rho g}{E}\left(\frac{y^2}{2} - hy\right) + g(x)$$

$\gamma_{xy} = 0 = -\dfrac{\nu \rho g x}{E} + f'(y) + g'(x) = 0$

$f'(y) = 0$; $f(y) = C_1$

$$g(x) = \frac{\nu g \rho x^2}{2E} + C_2$$

$$u = \frac{\nu \rho g}{E}(h-y)x \qquad \text{as } u=0 \text{ @ } x=0$$

$$v = \frac{\rho g}{E}\left(\frac{y^2}{2} - hy\right) + \frac{\nu \rho g x^2}{2E} \qquad \Big[\, C_2 = 0 \text{ as } v=0 \text{ @ } x=0 \,\&\, y=0$$

(c) $$U = \int_0^h \frac{1}{2}\sigma_{yy}\varepsilon_{yy}\, dy\, A$$

$$= \frac{A}{2E}\int_0^h \rho^2 g^2 (y-h)^2\, dy = \frac{A\rho^2 g^2 h^3}{6E}$$

(d) $$\varepsilon_{xx} = -\frac{\nu\sigma_{yy}}{E} + \alpha\Delta T$$

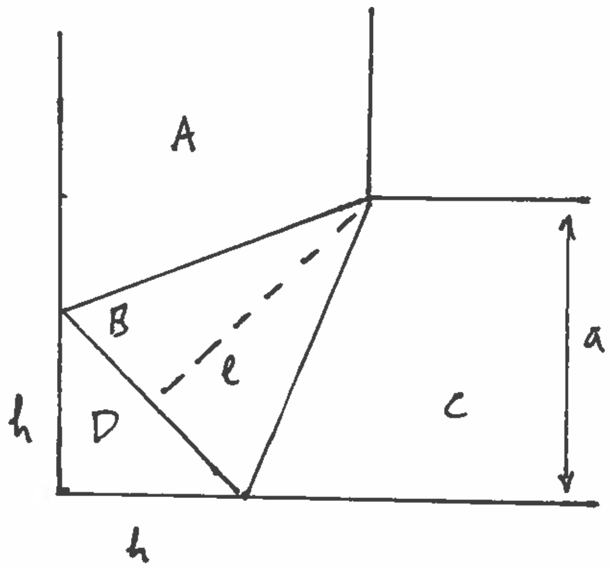
$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} + \alpha\Delta T$$

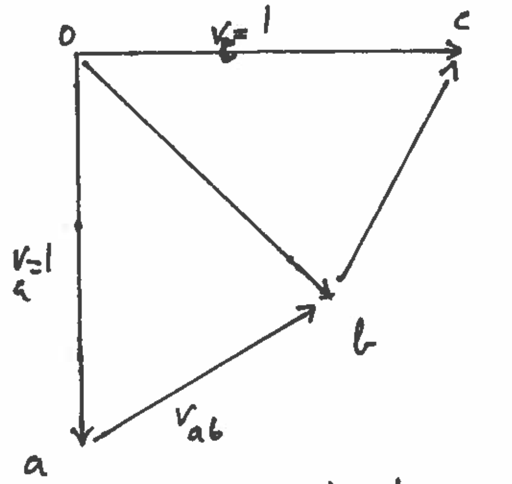
$$\Rightarrow u = \frac{\nu\rho g}{E}(h-y)x + \alpha\Delta T x$$

$$v = \frac{\rho g}{E}\left(\frac{y^2}{2} - hy\right) + \frac{\nu\rho g x^2}{2E} + \alpha\Delta T y$$

3 a) After postulating a collapse mechanism the maximum forming load can be computed. This gives a conservative estimate for the design of the forming equipment.

b)





velocity diagramm

$$l^2 = \sqrt{2}\,a - \frac{h}{\sqrt{2}}$$

- Volume conservation (set $v_a = 1$ too shorten the equations)

$V_a = V_c = 1$

$$1 \cdot a = v_b \cdot l \Rightarrow v_b = \frac{a}{\sqrt{2}\,a - \frac{h}{\sqrt{2}}} = \frac{1}{\sqrt{2} - \frac{x}{\sqrt{2}}} = \frac{\sqrt{2}}{2-x}$$

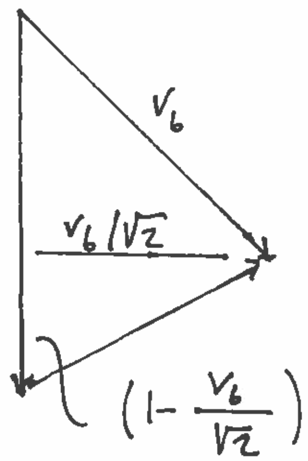
- determine v_{ab} from velocity diagramm

$$v_{ab}^2 = \frac{v_b^2}{2} + \left(1 - \frac{v_b}{\sqrt{2}}\right)^2$$

$$= \frac{v_b^2}{2} + 1 - \frac{2}{\sqrt{2}} v_b + \frac{v_b^2}{2}$$

$$= 1 - \sqrt{2}\, v_b + v_b^2$$

$$= 1 - \frac{2}{2-x} + \frac{2}{(2-x)^2} = \frac{x^2 - 2x + 2}{(2-x)^2}$$



- length of interface between blocks A and B

$$l_{ab}^2 = a^2 + (a-h)^2 = 2a^2 - 2ah + h^2$$

- length of interface between blocks B and D

$$l_{bd} = \sqrt{2}\, h$$

<u>Upper bound theorem</u>

$$p \cdot a \cdot 1 = k\,(l_{ab}\, v_{ab} + l_{bd}\, v_{bd} + l_{bc}\, v_{bc})$$

note that $l_{ab} = l_{bc}$

$v_{ab} = v_{bc}$

$v_{bd} = v_{b}$

$\Longrightarrow$

$$\frac{pa}{k} = 2\sqrt{2a^2 - 2ah + h^2} \cdot \frac{\sqrt{x^2 - 2x + 2}}{2-x} + \sqrt{2}h\,\frac{\sqrt{2}}{2-x}$$

$$\frac{p}{2k} = \sqrt{2 - 2x + x^2}\;\frac{\sqrt{x^2 - 2x + 2}}{2-x} + \frac{x}{2-x}$$

$$= \frac{x^2 - x + 2}{2-x} = 1 + \frac{x^2}{2-x}$$

ii) $$\frac{d(p/2k)}{dx} = \frac{2x}{2-x} + \frac{x^2}{(2-x)^2} = 0$$

$$2x(2-x) + x^2 = 0$$

$$4x - x^2 = 0 \Longrightarrow x = 0\ ;\ p_{min} = 2k$$

iii)

Length of "outer lining": $2(a + a - h) = 4a - 2h$

Length of "inner lining" $= 2a$

<u>Upper bound theorem</u>

$$p \cdot a = 2ka\left(1 + \frac{x^2}{2-x}\right) + kf\,(4a - 2h + 2a)$$

$$\frac{p}{2k} = 1 + \frac{x^2}{2-x} + f\,(3 - x)$$

(4)

$$\phi = \frac{Cr^2}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

$$\frac{\partial\phi}{\partial r} = \frac{2Cr}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

$$\frac{\partial^2\phi}{\partial r^2} = \frac{2C}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

$$\frac{\partial\phi}{\partial\theta} = \frac{Cr^2}{\tan\alpha - \alpha}\left[-1 + \cos 2\theta + 2\tan\alpha\sin\theta\cos\theta\right]$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \frac{Cr^2}{\tan\alpha - \alpha}\left[-2\sin 2\theta + 2\tan\alpha\left(\cos^2\theta - \sin^2\theta\right)\right]$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] = \frac{C}{\tan\alpha - \alpha}\left[-1 + \cos 2\theta + 2\tan\alpha\sin\theta\cos\theta\right]$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$= \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\theta - \sin 2\theta - 2\tan\alpha\sin^2\theta\right]$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\theta + \sin 2\theta - 2\tan\alpha\cos^2\theta\right]$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] = \frac{C}{\tan\alpha - \alpha}\left[1 - \cos 2\theta - 2\tan\alpha\sin\theta\cos\theta\right]$$

ii) along OB : $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$

along OA : $\sigma_{r\theta} = 0$; $\sigma_{\theta\theta} = -p$

$$\sigma_{\theta\theta}\Big|_{\theta=\alpha} = \frac{c}{\tan\alpha - \alpha}\left[2\alpha - 2\alpha - \sin 2\alpha - 2\sin\alpha\cos\alpha\right] = 0$$

$$\sigma_{r\theta}\Big|_{\theta=\alpha} = \frac{c}{\tan\alpha - \alpha}\left[1 - \cos 2\alpha - 2\sin^2\alpha\right] = 0$$

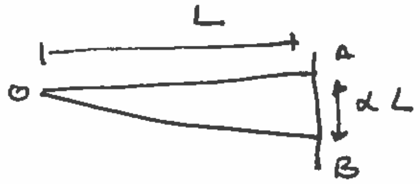
$\Rightarrow$ B. C. along OB satisfied

$$\sigma_{\theta\theta}\Big|_{\theta=0} = \frac{c}{\tan\alpha - \alpha}\left[2\alpha - 2\tan\alpha\right] = -2c = -p$$

$$c = p/2$$

$$\sigma_{r\theta}\Big|_{\theta=0} = \frac{c}{\tan\alpha - \alpha}\left[1 - 1 - 0\right] = 0$$

(c) (i)



consider cantilever of unit depth

$$M = pL\,\frac{L}{2} = \frac{pL^2}{2}$$

$$I \text{ @ } AB = \frac{bd^3}{12} = \frac{1(\alpha L)^3}{12}$$

$$\sigma \text{ at } A = \frac{My}{I} = \frac{pL^2}{2}\,\frac{\alpha L}{2}\,\frac{12}{\alpha^3 L^3} = \frac{3p}{\alpha^2}$$

(ii)

$$\sigma_{rr}\Big|_{\theta=0} = \frac{p}{2(\tan\alpha - \alpha)}(2\alpha)$$

$$\tan\alpha \sim \alpha + \frac{\alpha^3}{3}$$

$$= \frac{p}{\frac{2\alpha^3}{3}} 2\alpha = \frac{3p}{\alpha^2}$$