ENGINEERING TRIPOS PART IIA

2016 - 2017, PAPER 3C7

MECHANICS OF SOLIDS

3C7 Crib 2016/17

Q1.

(a) Airy stress function iff $\nabla^4 \phi = 0$

$\Rightarrow$ $4Ay + 6Dy = 0$, ie $\underline{D = -\frac{2A}{3}}$

No requirement on B & C.

(b) $\sigma_{xx} = A\left(x^2 y - \frac{2y^3}{3}\right)$; $\sigma_{yy} = Ay^3 + 2Bxy + Cy$

$\sigma_{xy} = -Axy^2 - Bx^2 - Cx$

B C.

$\sigma_{xy}(x, y = \pm b) = 0 = -Axb^2 - Bx^2 - Cx \Big] \Rightarrow B = 0$, $C = -Ab^2$

$-\sigma_{yy}(x, y = -b) = q = \frac{Ab^3}{3} + 2Bxb + Cb$

$\sigma_{yy}(x, y = +b) = q = \frac{Ab^3}{3} + 2Bxb + Cb$

$\Rightarrow$ $B = 0$, $C = -Ab^2$, $A = -\frac{3q}{2b^3}$; $D = \frac{q}{b^3}$

$\Rightarrow C = \frac{3q}{2b}$

Stresses

$\sigma_{xx} = -\frac{3q}{2b^3}\left(x^2 y - \frac{2y^3}{3}\right)$; $\sigma_{yy} = -\frac{3q}{2b^3}\left(y^3 - yb^2\right)$

$\sigma_{xy} = \frac{3q}{2b^3}\left(-b^2 x + xy^2\right)$

(c) $S(x) = \int_{-b}^{b} \sigma_{xy}\,dy = -2qx$ consistent with equilibrium. $S(x) = \int_{-b}^{b} \frac{3q}{2b^3}\, x\,(y^2-b^2)\,dy$

(d) $M(0) = \int_{-b}^{b} \sigma_{xx}(0,y)\,y\,dy = \frac{2b^2q}{5}$

$$M(0) = \int_{-b}^{b} \frac{3q}{2b^3}\,\frac{2y^4}{3}\,dy = \frac{2}{5}b^2 q$$

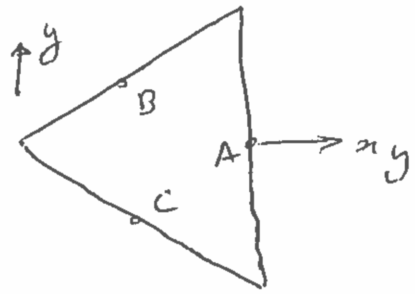
Thus add a constant bending moment to correct. Only ~~a~~ modifies σ_{xx}

$$\sigma_{xx} = \frac{3q}{2b^3}\left(x^2y - \frac{2y^3}{3}\right) - \underbrace{\frac{2b^2q}{5}\,y\,\overbrace{\left(\frac{3}{2b^3}\right)}^{I}}_{\Delta\sigma_{xx}}$$

$$= \frac{3q}{2b^3}\left(x^2y - \frac{2y^3}{3} - \frac{2b^2y}{5}\right)$$

$$\Delta\sigma_{xx} = -\frac{M(0)\,y}{I} = -\frac{3}{5}\,\frac{qy}{b}$$

(2.)



(a) $\phi = -\frac{G\alpha}{2}\left(\eta y^2 - x^2\right)\left(1-\frac{x}{a}\right)$

$\phi = 0$ on boundary

(i) boundary $x = a$ $\Rightarrow$ satisfied

(ii) $y = \pm \frac{x}{\sqrt{3}}$

$$\eta\frac{x^2}{3} - x^2 = 0 \Rightarrow \eta = 3$$

$$\therefore \phi = -\frac{G\alpha}{2}\left(3y^2 - x^2\right)\left(1-\frac{x}{a}\right)$$

$$\nabla^2\phi = ? = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{G\alpha}{2}\left(-2 + \frac{6x}{a} - \frac{6x}{a} + 6\right)$$

$$= -2G\alpha \checkmark$$

(b) $T = 2\int \phi \, dA$

$$T = \frac{-4G\alpha}{2}\int_0^a\left[\int_0^{x/\sqrt{3}}\left(3y^2 - 3y^2\frac{x}{a} - x^2 + \frac{x^3}{a}\right)dy\right]dx$$

$$= \frac{2a^4 G\alpha}{30\sqrt{3}} = \frac{\sqrt{3}\,a^4 G\alpha}{45}$$

(C) Max. shear at A, B, C. By symmetry all equal

$$\Rightarrow \sigma_{xy}(a,0) = -\frac{\partial \phi}{\partial x}(a,0) = \frac{G\alpha}{2}\left(-\frac{3y^2}{a} - 2x + \frac{3x^2}{a}\right)$$

$$= \frac{G\alpha a}{2}$$

$$= \frac{15\sqrt{3}}{2}\frac{T}{a^3}$$

(d)

(i) At corners since no shear stress along each edge at corners

(ii) By symmetry at centroid of Δ.

3. (a) (i)

Isotropic as f is defined in terms of the stress invariants I_1, $\bar{\sigma}$.
Pressure sensitive because f depends upon the hydrostatic stress $\sigma_h = \frac{1}{3} I_1$.

3. (a) (ii) $f = \bar{\sigma}^2 + (c_1 - c_2) I_1 - c_1 c_2 = 0$

$I_1 = \sigma_1 + \sigma_2 + \sigma_3$

$2\bar{\sigma}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

Tension test: $\sigma_1 = \sigma_t \Rightarrow I_1 = \sigma_t, \bar{\sigma} = \sigma_t$

$$f = \sigma_t^2 + (c_1 - c_2)\sigma_t - c_1 c_2 = 0 \quad (1)$$

Compression test: $\sigma_1 = -\sigma_c \Rightarrow I_1 = -\sigma_c$

$\bar{\sigma} = \sigma_c$

$$\Rightarrow f = \sigma_c^2 - (c_1 - c_2)\sigma_c - c_1 c_2 = 0 \quad (2)$$

subtract: $\sigma_t^2 - \sigma_c^2 + (c_1 - c_2)(\sigma_t + \sigma_c) = 0$

$\Rightarrow \sigma_t - \sigma_c + c_1 - c_2 = 0$

$$\Rightarrow c_1 = c_2 + \sigma_c - \sigma_t \quad (3)$$

Substitute into (1) to get a quadratic equation in c_2:

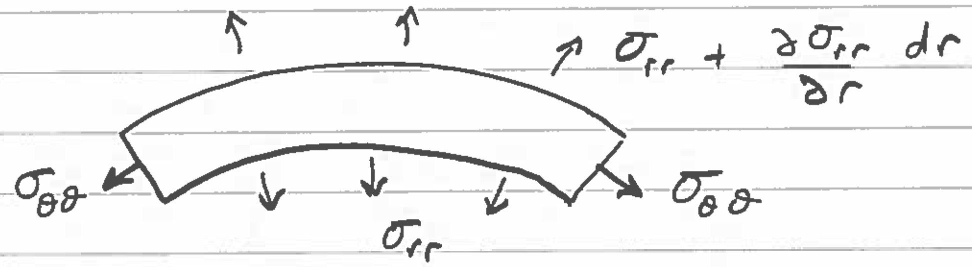
$$\left(c_2 + \frac{\sigma_c - \sigma_t}{2}\right)^2 = \frac{1}{4}(\sigma_c + \sigma_t)^2$$

$$\Rightarrow c_2 + \frac{\sigma_c - \sigma_t}{2} = \pm \frac{1}{2}(\sigma_c + \sigma_t)$$

$\Rightarrow c_2 = \sigma_t, \quad c_1 = \sigma_c$

or $c_2 = -\sigma_c, \quad c_1 = -\sigma_t.$

3. (b) (i)



Equilibrium in radial direction :

$$(r + dr)\, d\theta \left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} dr \right) - r\, d\theta\, \sigma_{rr}$$

$$- 2 \sigma_{\theta\theta}\, dr \frac{d\theta}{2} = 0$$

$$\Rightarrow \quad \sigma_{rr} + r \frac{d\sigma_{rr}}{dr} - \sigma_{\theta\theta} = 0$$

(ii) $\sigma_{rr} = A - \frac{B}{r^2}$ $\qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}$

BC : $A - \frac{B}{b^2} = -p \qquad A - \frac{B}{a^2} = 0$

$$\Rightarrow \quad A = B/a^2$$

$$\Rightarrow \quad A = \frac{b^2}{a^2 - b^2} p \qquad B = \frac{a^2 b^2}{a^2 - b^2} p$$

3 (b) (ii)

$$B = \frac{b^2}{1-m^2}\, p \qquad A = \frac{m^2}{1-m^2}\, p \qquad m \equiv \frac{b}{a}$$

$$\sigma_{rr} = p\frac{m^2}{1-m^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = p\frac{m^2}{1-m^2}\left(1 + \frac{a^2}{r^2}\right)$$

σ_{zz} from equilibrium :

$$-p\,\pi b^2 = \sigma_{zz}\left(\pi b^2 - \pi a^2\right)$$

$$\Rightarrow \qquad \sigma_{zz} = \frac{m}{1-m}\cdot p$$

(iii) Yield initiated at inner surface.

$$\sigma_{rr} - \sigma_{\theta\theta} = Y \qquad \Rightarrow \qquad -\frac{2B}{a^2} = Y$$

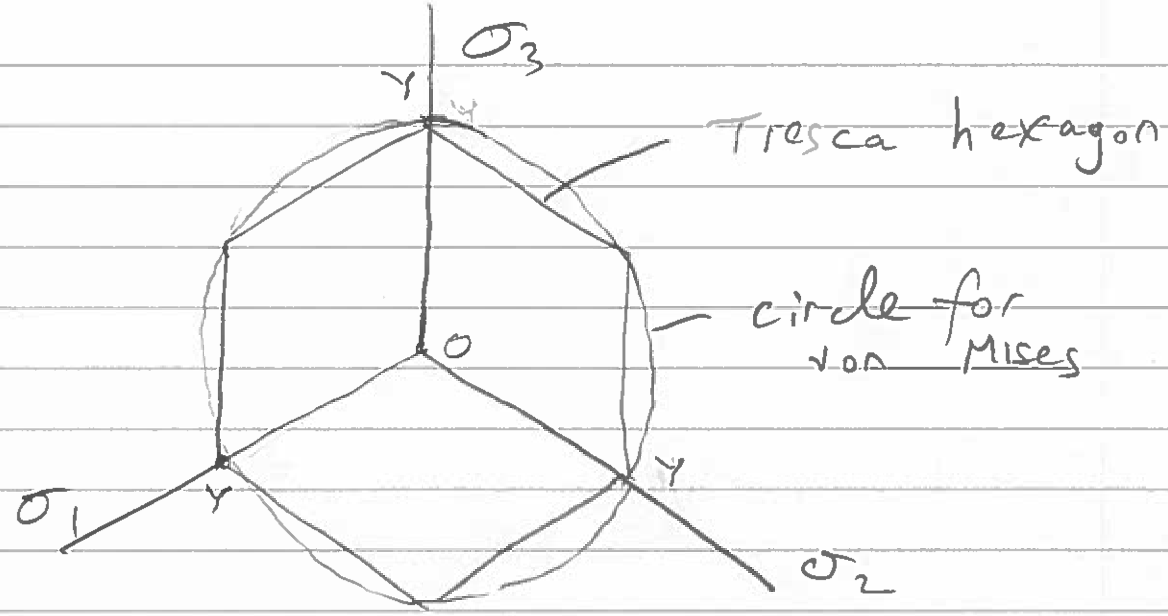
$$\Rightarrow \quad -\frac{m^2}{1-m^2}\, p_e = \frac{Y}{2} \qquad \Rightarrow \qquad p_e = \frac{Y}{2}\left(\frac{m^2-1}{m^2}\right)$$

Entire cross-section is plastic :

$$\frac{-(\sigma_{rr} - \sigma_{\theta\theta})}{r} = \frac{d\sigma_{rr}}{dr}$$

$$\Rightarrow \quad -\int_0^{-p_u} d\sigma_{rr} = Y\int_a^b \frac{dr}{r} \qquad \Rightarrow \qquad \frac{p_u}{Y} = \ln m$$

4, (a)



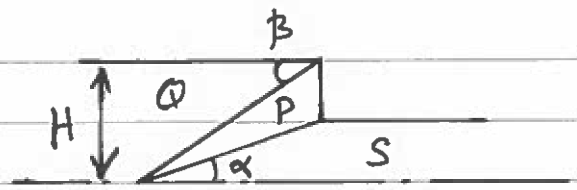
Look down the hydrostatic axis: ~~the~~ yield surface is a hexagonal prism for Tresca & a circular cylinder for von Mises.

4. (b) Kinematics + Yield ⇒ upper bound
Equilibrium + Yield ⇒ Lower bound

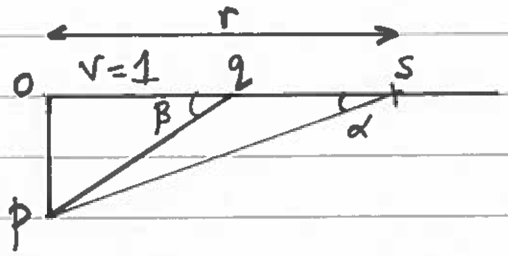
Upper bound: assume a velocity field, & calculate the plastic dissipation. Gives an upper bound on ~~the~~ collapse load.

Lower bound: assume an equilibrium stress field ~~that~~ nowhere violates yield. Gives a lower bound on ~~the~~ collapse load.

4. (c)



Extrusion ratio
$r = H/h$



Hodograph

$$V_{oq} = 1 \qquad V_{os} = \frac{H}{h} = r$$

$$V_{op} = \tan\alpha \qquad V_{pq}^2 = 1 + r^2\tan^2\alpha$$

$$V_{ps} = \frac{r}{\cos\alpha}$$

$$l_{ps} = \frac{h}{\sin\alpha} \qquad l_{pQ} = \frac{H}{\sin\beta}$$

$$\sin\beta = \frac{V_{op}}{V_{pq}} = \frac{r\tan\alpha}{(1+r^2\tan^2\alpha)^{1/2}} \Rightarrow l_{pQ} = \frac{H(1+r^2\tan^2\alpha)^{1/2}}{r\tan\alpha}$$

Power : $$1 \cdot pH = kH\frac{(1+r^2\tan^2\alpha)^{1/2}}{r\tan\alpha}(1+r^2\tan^2\alpha)^{1/2} + \frac{khr}{\sin\alpha\cos\alpha}$$

$$\Rightarrow \quad \underline{p = k(r+1)\left(\tan\alpha + \frac{1}{r\tan\alpha}\right)}$$

Now minimise p with respect to α.
write $x = \tan\alpha$

$$p = k(r+1)\left(x + \frac{1}{rx}\right)$$

$$\frac{dp}{dx} = 0 \Rightarrow 1 - \frac{1}{rx^2} = 0 \Rightarrow x = \frac{1}{\sqrt{r}}$$

$$\Rightarrow p_{min} = k(r+1)\frac{2}{\sqrt{r}}, \quad r = H/h$$

(ii) Power in $= pV2H$ in extrusion.

Power dissipated $= CkV$ where $C =$ constant

$$\Rightarrow pV2H = CkV \Rightarrow C = \frac{p.2H}{k}$$

In drawing operation, power in $= t.\frac{VH}{h}.2h$

power dissipated $= CkV$ as before.

Hence $t.\frac{VH}{h}2h = CkV = \frac{p}{k}2H\,kV$

$$\Rightarrow t = p.$$