3C7 2019

CRIB

1a) Strains must be compatible

for $\varepsilon_{xx} = A_1 x + A_2 xy + A_3 y^2$

$\varepsilon_{yy} = B_1 + B_2 x + B_3 y$

$\gamma_{xy} = C_1 x + C_2 y + C_3 xy$

i.e. $$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

Calculate quantities

$$\frac{\partial \varepsilon_{xx}}{\partial y} = A_2 x + 2A_3 y \; ; \; \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = 2A_3$$

$$\frac{\partial \varepsilon_{yy}}{\partial x} = B_2 \; ; \; \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 0$$

$$\frac{\partial \gamma_{xy}}{\partial x} = C_1 + C_3 y \; ; \; \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = C_3$$

Compatibility implies $C_3 = 2A_3$

(b) Must integrate to find displacements.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\Rightarrow u = \frac{A_1}{2} x^2 + \frac{A_2}{2} x^2 y + A_3 y^2 x + f(y)$$

1(b) cont.

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad v = B_1 y + B_2 xy + \frac{B_3}{2} y^2 + g(x)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{A_2}{2} x^2 + 2A_3 xy + f'(y) \\ \frac{\partial v}{\partial x} &= B_2 y + g'(x) \end{aligned}\right\} \text{ give conditions on } \gamma_{xy}$$

sub into

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \text{using } C_3 = 2A_3$$

$$C_1 x + C_2 y + \cancel{2A_3 xy} = \frac{A_2}{2} x^2 + \cancel{2A_3 xy} + B_2 y + f'(y) + g'(x)$$

Gather terms

$$\left.\begin{aligned} g'(x) &= C_1 x - \frac{A_2 x^2}{2} + D \\ f'(y) &= C_2 y - B_2 y - D \end{aligned}\right\} \text{unknown constant}$$

Very few candidates included this term.

Integrate

$$\left.\begin{aligned} g(x) &= C_1 \frac{x^2}{2} - \frac{A_2}{6} x^3 + Dx + E \\ f(y) &= \frac{1}{2}(C_2 - B_2) y^2 - Dy + F \end{aligned}\right\} \text{unknown constants}$$

Giving

$$u = \frac{A_1}{2} x^2 + \frac{A_2}{2} x^2 y + A_3 y^2 x + \frac{1}{2}(C_2 - B_2) y^2 - Dy + F$$

$$v = B_1 y + B_2 xy + \frac{B_3}{2} y^2 + \frac{C_1}{2} x^2 - \frac{A_2}{6} x^3 + Dx + E$$

(c) Consider the origin

$$u = F$$
$$v = E$$

$$\text{Rotation } \Theta = \left. \frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{2} \right|_{x=y=0} = D$$

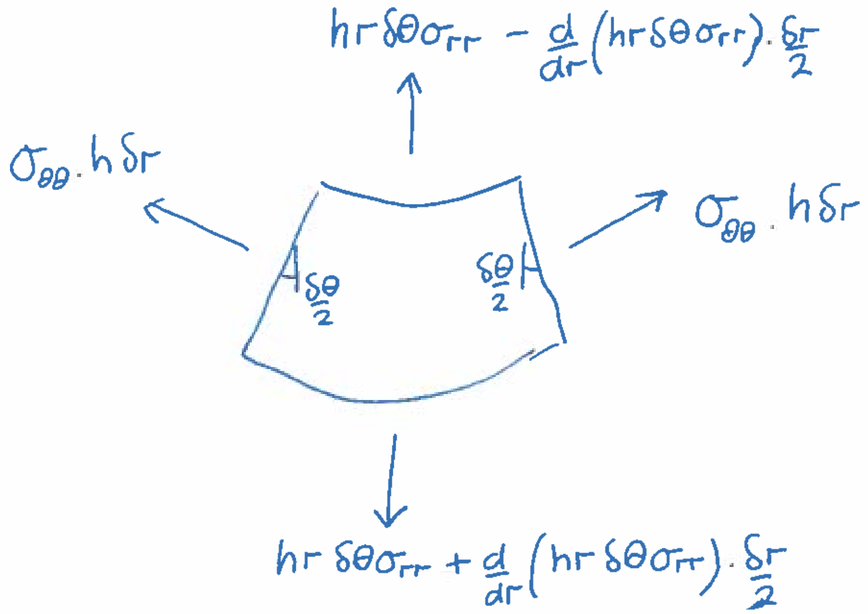
Thus F, E, D represent the displacement of the origin in the x, y direction, and the rotation of the origin, respectively. They can only be fixed by boundary conditions on the displacements

(d) If the origin is fixed $\Rightarrow$ $E=0,\ F=0$

If the x-axis is fixed in the y-direction, it implies that none of the terms in v can depend on x when $y=0$

$$\Rightarrow \quad C_1 = A_2 = D = 0$$

2(a) Consider an element centered at r with dimensions $\delta\theta$, δr. Show forces (including first order variation) on faces



Radial equilibrium of forces

$$\delta\theta\,\delta r\,\frac{d}{dr}(hr\sigma_{rr}) = 2\,\sigma_{\theta\theta}.h.\delta r\,\frac{\delta\theta}{2}$$

$$\frac{d(hr\sigma_{rr})}{dr} = \sigma_{\theta\theta}.h$$

or

$$r.\frac{d(h\sigma_{rr})}{dr} + h\sigma_{rr} = h\sigma_{\theta\theta}$$

$$\Rightarrow \quad \frac{d(h\sigma_{rr})}{dr} = \frac{h}{r}(\sigma_{\theta\theta} - \sigma_{rr}) \quad \checkmark$$

(b) For an elastic soln., we require equilibrium & compatibility for $\sigma_{rr} = A - \frac{B}{r}$; $\sigma_{\theta\theta} = \frac{B}{r}$

Equilibrium $\sigma_{\theta\theta} - \sigma_{rr} = \frac{2B}{r} - A$; $\frac{d(h\sigma_{rr})}{dr} = \frac{d}{dr}\left(\frac{Ha}{r}A - \frac{Ha}{r^2}\cdot B\right)$

$$= -\frac{HaA}{r^2} + \frac{2HaB}{r^3}$$

sub in equilibrium equation

$$Ha\left(-\frac{A}{r^2} + \frac{2B}{r^3}\right) = \frac{Ha}{r^2}\left(\frac{2B}{r} - A\right)$$ ✓

Very few candidates considered this

Compatibility

With no thermal strains or body forces, need to satisfy

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}\left\{\sigma_{rr} + \sigma_{\theta\theta}\right\} = 0$$

Satisfied as $\sigma_{rr} + \sigma_{\theta\theta} = A$ = constant

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(c) Boundary condition at $r = b$, $\sigma_{rr} = 0$

$\Rightarrow 0 = A - \frac{B}{b}$, $A = \frac{B}{b}$ (1)

at $r = a$, $\sigma_{rr} = -p$

$\Rightarrow -p = A - \frac{B}{a}$ (2)

(1) & (2) give $A = \frac{ap}{b-a}$; $B = \frac{abp}{b-a}$

$\sigma_{rr} = \frac{pa}{(b-a)}\left[1 - \frac{b}{r}\right]$; $\sigma_{\theta\theta} = \frac{pa}{(b-a)}\left[\frac{b}{r}\right]$

(always ≤ 0) (always > 0)

$\sigma_{33} = 0$ (plane strain), so yield is always governed by $\sigma_{\theta\theta} - \sigma_{rr} = Y \Rightarrow \dfrac{pa}{(b-a)}\left[\dfrac{2b}{r} - 1\right] = Y$

worst case (initial yield) at $r = a$

$$\Rightarrow p = \frac{Y(b-a)}{a} \cdot \frac{a}{(2b-a)} = \frac{Y(b-a)}{(2b-a)}$$

(d) Assume $\sigma_{\theta\theta} - \sigma_{rr} = Y$ everywhere

Equilibrium gives

$$\frac{d(h\sigma_{rr})}{dr} = \frac{Ha}{r^2} \cdot Y$$

Integrate between $r = a$, where $h\sigma_{rr} = -Hp$ and $r = b$, where $h\sigma_{rr} = 0$ $\Big\}$ B.C. on stress

$$\int_{-Hp}^{0} d(h\sigma_{rr}) = Y \int_a^b \frac{Ha}{r^2}\, dr$$

$$\left[h\sigma_{rr}\right]_{-Hp}^{0} = HaY\left[-\frac{1}{r}\right]_a^b$$

$$p = Y\left(1 - \frac{a}{b}\right)$$

3.
$$\phi = \frac{W}{20Lbd^3}\left[20x^2y^3 - 15d^2x^2y - 4y^5 - 5d^3x^2 - y^3(5L^2-2d^2)\right]$$

$$\frac{\partial\phi}{\partial x} = \frac{W}{20Lbd^3}\left[40xy^3 - 30d^2xy - 10d^3x\right]$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{W}{20Lbd^3}\left[40y^3 - 30d^2y - 10d^3\right]$$

$$\frac{\partial\phi}{\partial y} = \frac{W}{20Lbd^3}\left[60x^2y^2 - 15d^2x^2 - 20y^4 - 3y^2(5L^2-2d^2)\right]$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \frac{W}{20Lbd^3}\left[120x^2y - 80y^3 - 6y(5L^2-2d^2)\right]$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{W}{20Lbd^3}\left[120xy^2 - 30d^2x\right]$$

(a) At $y = \frac{d}{2}$, check stresses directly

$$\sigma_{yy} = \frac{W}{20Lbd^3}\left[5d^3 - 15d^3 - 10d^3\right] = -\frac{W}{Lb} \checkmark$$

$$\sigma_{xy} = \frac{-W}{20Lbd^3}\left[30xd^2 - 30d^2x\right] = 0 \checkmark$$

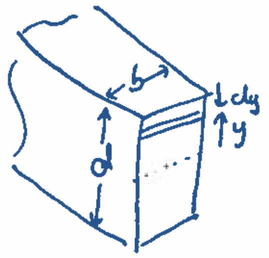
At $y = -\frac{d}{2}$, check stresses directly.

$$\sigma_{yy} = \frac{W}{20Lbd^3}\left[-5d^3 + 15d^3 - 10d^3\right] = 0 \checkmark$$

$$\sigma_{xy} = 0 \quad (\sigma_{xy} \text{ is even in } y) \checkmark$$

3 cont (b) Must satisfy boundary conditions for the stress resultants at $x = \pm \frac{L}{2}$ (tension, shear force, bending moment)

Consider an element



Tension $$T = \int_{-d/2}^{d/2} \sigma_{xx} \cdot b \cdot dy$$

$= 0$ for all x (clear as σ_{xx} only contains odd powers of y) ✓

Shear force $$S = \int_{-d/2}^{d/2} \sigma_{xy} \cdot b \cdot dy$$

$$= -\frac{W}{20Ld^3}\left[40xy^3 - 30d^2xy\right]_{y=-d/2}^{y=+d/2}$$

$$= -\frac{W}{20} \cdot \frac{x}{L} \cdot \frac{1}{d^3}\left(40\frac{d^3}{8} - \frac{30d^3}{2}\right) \times 2$$

$$= \frac{Wx}{L}$$

At $x = \pm \frac{L}{2}$, $S = \pm \frac{W}{2}$ ✓

Bending moment
$$M = \int_{-d/2}^{d/2} \sigma_{xx} \cdot y \cdot b\,dy$$

$$M = \frac{W}{20} \cdot \frac{1}{Ld^3} \cdot \left[40x^2y^3 - 16y^5 - 2y^3(5L^2 - 2d^2)\right]_{y=-d/2}^{y=+d/2}$$

$$= \frac{W}{20} \cdot \frac{1}{Ld^3}\left(40x^2\frac{d^3}{8} - 10L^2\frac{d^3}{8}\right) \times 2$$

$$= \frac{W}{2} \cdot \frac{1}{L}\left(x^2 - \left(\frac{L}{2}\right)^2\right)$$

3(b) cont.

at $x = \pm\frac{L}{2}$, $M = 0$ (simply supported) ✓

[Check, at $x=0$, $M = -\frac{WL}{8}$, as required by equilibrium]

(c) At $x=0$, $y=\frac{d}{2}$

$$\sigma_{xx} = \frac{W}{20Lbd^3}\left[-\frac{80d^3}{8} - 30L^2\frac{d}{2} + \frac{12d^3}{2}\right]$$

$$= \frac{W}{20Lbd^3}\left[-4d^3 - 15L^2d\right]$$

$$= \frac{-W}{20Lb}\left[15\left(\frac{L}{d}\right)^2 + 4\right] \qquad (1)$$

Consider simple beam theory, with $M = -\frac{WL}{8}$, $\frac{\sigma}{y} = \frac{M}{I}$

$y = \frac{d}{2}$, $I = \frac{bd^3}{12}$

$$\Rightarrow \sigma = -\frac{WL}{8}\cdot\frac{d}{2}\cdot\frac{12}{bd^3} = \frac{-W}{20Lb}\left[15\left(\frac{L}{d}\right)^2\right] \qquad (2)$$

Compare (1) & (2). The ratio of the discrepancy (4) to the common term $\left(15\left(\frac{L}{d}\right)^2\right)$ must be $\frac{1}{100} = 1\%$

$$4 = \frac{1}{100}\cdot 15\left(\frac{L}{d}\right)^2$$

$$\Rightarrow \left(\frac{L}{d}\right) = 5.16$$

This will be the least value required for 1% discrepancy.

4(a) For a ductile, elasto-plastic material:

(i) If a collapse load is found by equating the work done by the load with the energy dissipated in yield for any compatible displacement field, that will be an upper bound on the collapse load.

(ii) If a stress field can be found in the solid that is in equilibrium with the applied load, and nowhere violates the yield condition, the applied load will be a lower bound on the collapse load.

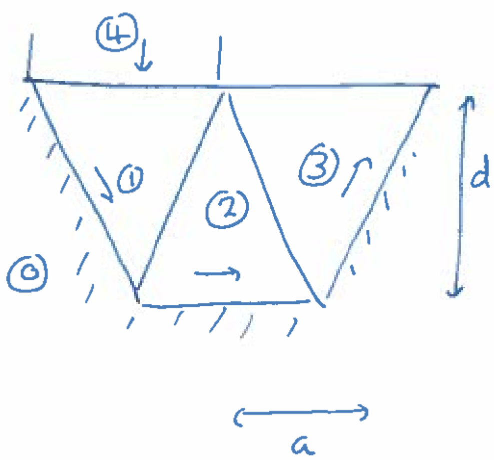
This was, in general, poorly answered. Accurate statements of the theorems were rare.

Upper bound solutions do not require equilibrium to be satisfied.
Lower bound solutions do not require compatibility to be satisfied.

[This question is an upper bound question]

(b) Assume question requires force F PER UNIT DEPTH into page.
Take unit depth.
Calculate a displacement field for small displacements (or, equivalently, consider velocities)

Only consider half of the problem - the left side will be a symmetric copy.



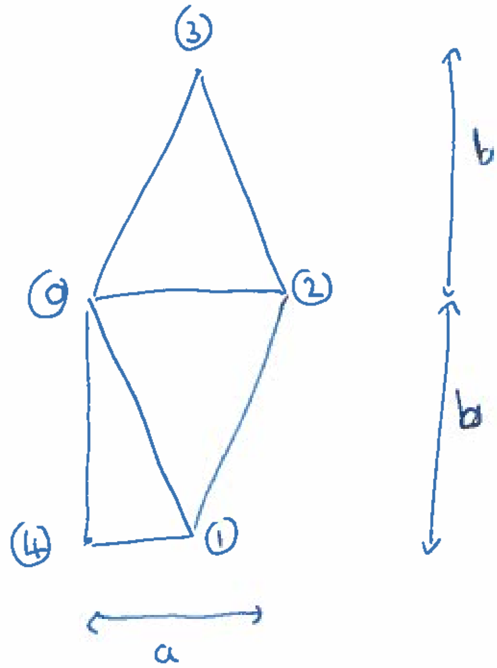
Undisturbed ground labelled (0)
Blocks of soil labelled (1), (2), (3)
Footing labelled (4)

l_{ab} is length of boundary between (a) & (b)

4(b) cont.

Draw a displacement diagram with block (4) moving down δ [scaling $\frac{\delta}{b}$ applied so drawing shows similarly with physical geometry]

$\left(\frac{\delta}{b}\right) \times$



For convenience, define $l^2 = b^2 + \left(\frac{a}{2}\right)^2$

δ_{ab} is length of line between (a) & (b)

For all calculation, consider unit depth into page

(i) Work done $= \frac{F}{2} \cdot \delta$

Energy Dissipated $= k\left(l_{01} \cdot \delta_{01} + l_{02}\delta_{02} + l_{03}\delta_{03} + l_{12}\delta_{12} + l_{23}\delta_{23}\right)$

$$= \frac{\delta k}{b}\left(l^2 + a^2 + l^2 + l^2 + l^2\right)$$

$$= \frac{\delta k}{b}\left(4l^2 + a^2\right) = \frac{\delta k}{b}\left(4b^2 + 2a^2\right)$$

Equate work done & energy dissipated for an upper bound soln.

$$\frac{F}{2}\delta = \frac{\delta k}{b}\left(4b^2 + 2a^2\right)$$

$$\therefore F = 2ak\left(4\left(\frac{b}{a}\right) + 2\left(\frac{a}{b}\right)\right)$$ for any $\left(\frac{a}{b}\right)$ is an upper bound.

4(b)(i) cont.

For an optimum upper bound, find minimum F for varying $q = \left(\frac{a}{b}\right)$

$$F = 2ak\left(4q^{-1} + 2q\right)$$

$$\frac{dF}{dq} = 2ak\left(-\frac{4}{q^2} + 2\right) = 0 \text{ for minimum}$$

$$\Rightarrow q = \sqrt{2}$$

$$\underline{\underline{F_{\text{optimum}(i)} = 8\sqrt{2}ak}}$$

(ii) For rough interface, assume a slip line along the OH boundary with a flow stress k, and hence additional work done $= l_{14} S_{14} = a \cdot \frac{a}{2}\left(\frac{\delta}{b}\right) = \frac{\delta}{b}\left(\frac{a^2}{2}\right)$

Very few candidates explicitly noted this

Now, Total E.D. $= \frac{\delta k}{b}\left(4c^2 + a^2 + \frac{a^2}{2}\right)$

Equate E.D. & W.D

$$F = 2ak\left(4\left(\frac{b}{a}\right) + \frac{5}{2}\left(\frac{a}{b}\right)\right)$$

Find optimum

$$\frac{dP}{dq} = 2ak\left(-\frac{4}{q^2} + \frac{5}{2}\right) = 0$$

$$\therefore 5q^2 = 8;\quad q = \frac{2\sqrt{2}}{\sqrt{5}}$$

$$\underline{\underline{F_{\text{optimum}(ii)} = 4\sqrt{10}ak}} \qquad (\simeq 12\% \text{ greater than (i)})$$

4(b) cont.

(iii) Without special design features to ensure no slip, it is unlikely to be sensible to rely on the additional strength given by (ii). Even if no slip can be ensured, other mechanisms (perhaps rotating blocks) that do not generate slip should be explored. Hence the value in (i), $F = 8\sqrt{2}ak$, is a safer design choice.

Most candidates did not note that design needs to be conservative