PART IIA 2015 – MODULE 3C8 – SOLUTIONS

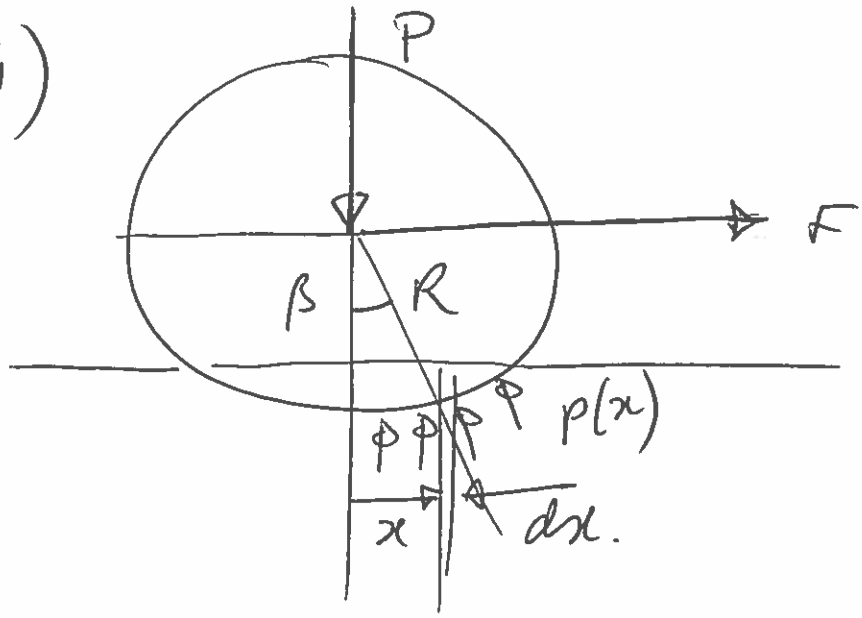
1 a) conditions – frictionless
elastic
small strains
non-conforming contact

b) i) $b = 2\sqrt{\dfrac{P'R}{\pi E^*}} \rightarrow \sqrt{\dfrac{E^*}{R}} = \dfrac{2}{b}\sqrt{\dfrac{P'}{\pi}}$

$$p_0 = \sqrt{\frac{P'E^*}{\pi R}}$$

$$\therefore\ p_0 = \frac{2}{b}\sqrt{\frac{P'}{\pi}}\sqrt{\frac{P'}{\pi}} = \underline{\underline{\frac{2P'}{b\pi}}}$$

ii)



$$P' = \int_0^b \cos\beta\, p(x)\,dx \qquad \text{small } \beta,\ \cos\beta \to 1$$

$$P' = \int_0^b 2p_0\left(1 - \frac{x^2}{b^2}\right)^{\frac{1}{2}} dx.$$

to account for load now being carried from $x=0$ to $x=b$ instead of $x=-b$ to $x=+b$.

$$F' = \int_0^b \sin\beta \; p(x)\,dx \qquad \text{where } \sin\beta = \frac{x}{R}$$

$$= \int_0^b \frac{x}{R}\, p(x)\,dx$$

$$= \frac{2p_0}{R}\int_0^b x\left(1 - \frac{x^2}{b^2}\right)^{\frac{1}{2}} dx$$

let $x = b\sin\theta$
$dx = b\cos\theta\, d\theta$

limits $\begin{cases} x = b, & \sin\theta = 1 \\ & \theta = \frac{\pi}{2} \\ x = 0, & \theta = 0 \end{cases}$

$$= \frac{2p_0}{R}\int_0^{\frac{\pi}{2}} b^2 \sin\theta\,(1-\sin^2\theta)^{\frac{1}{2}} \cos\theta\, d\theta$$

$$= \frac{2p_0 b^2}{R}\int_0^{\frac{\pi}{2}} \sin\theta \cos^2\theta\, d\theta$$

$$= \frac{2p_0 b^2}{R}\int_0^{\frac{\pi}{2}} \sin\theta\,(1-\sin^2\theta)\, d\theta$$

$$= \frac{2p_0 b^2}{R}\int_0^{\frac{\pi}{2}} \sin\theta - \frac{3}{4}\sin\theta + \frac{1}{4}\sin 3\theta\, d\theta$$

(using hint)

$$= \frac{2p_0 b^2}{R}\int_0^{\frac{\pi}{2}} \frac{1}{4}(\sin\theta + \sin 3\theta)\, d\theta$$

$$= \frac{2p_0 b^2}{4R}\left[-\cos\theta - \frac{\cos 3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2p_0 b^2}{4R}\left\{(-0-0) - \left(-1-\frac{1}{3}\right)\right\}$$

$$F' = \frac{2p_0 b^2}{3R} \qquad \text{but } p_0 = \frac{2P'}{b\pi} \qquad \therefore F = \frac{4\,P\,b}{3\pi\, R}$$

iii)

$$F = \frac{4}{3\pi} \frac{Pb}{R}$$

where $b \propto P^{\frac{1}{2}}$ (data book)

so $F \propto P^{\frac{3}{2}}$

$$\underline{\underline{\log_{10} F \propto \frac{3}{2} \log_{10} P.}}$$

from Fig 1b, slope is $\sim \dfrac{0.9-(-1)}{2.2-0.95}$

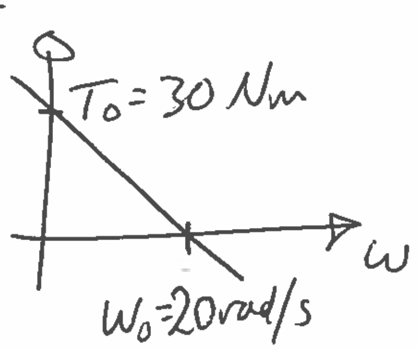
$$\underline{\underline{\sim 1.52}}$$

So very good agreement.

iv)

Some visco-elastic behaviour should be incorporated into the material property in order to generate loss of contact in the rear half of the contact patch. A more sophisticated model would account for shear deformation of the material.

2 . a) i)



$T = T_0 + m\omega \qquad m = \frac{-30}{20} = -\frac{3}{2} \frac{\text{Nms}}{\text{rad}}$

$P = T\omega = T_0\omega + m\omega^2$

$\frac{dP}{d\omega} = T_0 + 2m\omega = 0$

$\omega_{max} = \frac{-T_0}{2m} = \frac{30 \cdot 2}{2 \cdot 3} = 10$ rad/s

max power at $\omega_{max} = \underline{10 \text{ rad/s}}$

where $T = 15$ Nm

$\therefore$ max power $P_{max} = 10 \times 15 = \underline{\underline{150\,W}}$.

ii) max speed when $F \cdot V = P_{MAX} = 150\,W$.

$\therefore 0.4V^3 = 150$

$V^3 = \sqrt[3]{375}$

$V = 7.21$ m/s.

max power occurs at crank speed $\omega = 10$ rad/s

$V = \Omega R \qquad \Omega = G\omega$

$\therefore V = RG\omega$

$G = \frac{V}{\omega R} = \frac{7.21}{10 \cdot 0.33} = \underline{\underline{2.185}}$.

b) i) convert cyclist T, ω characteristic to F, V with $G = 3$:

$$V_0 = RG\omega_0 = 0{\cdot}33 . 3 . 20 = \underline{\underline{20 \text{ m/s}}}$$

$$F_0 = \frac{T_0}{RG} = \frac{30}{0{\cdot}33 . 3} = \underline{\underline{30 \text{ N.}}}$$

convert max motor torque T_{motor} to max traction force on road F_{motor}.

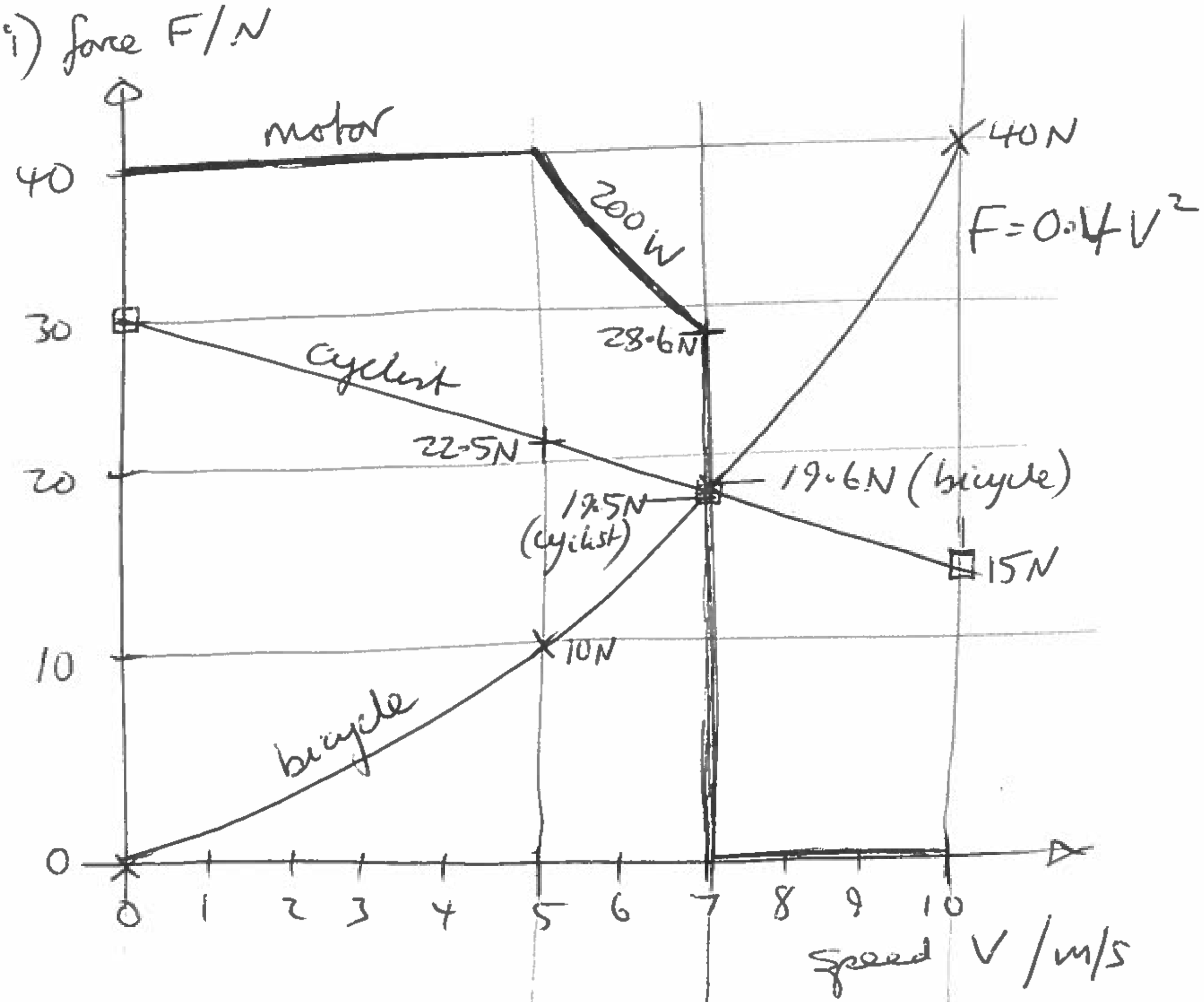
$$F_{motor} = \frac{T_{motor}}{R} = \frac{13{\cdot}3}{0{\cdot}33} = \underline{\underline{40 \text{ N}}}$$

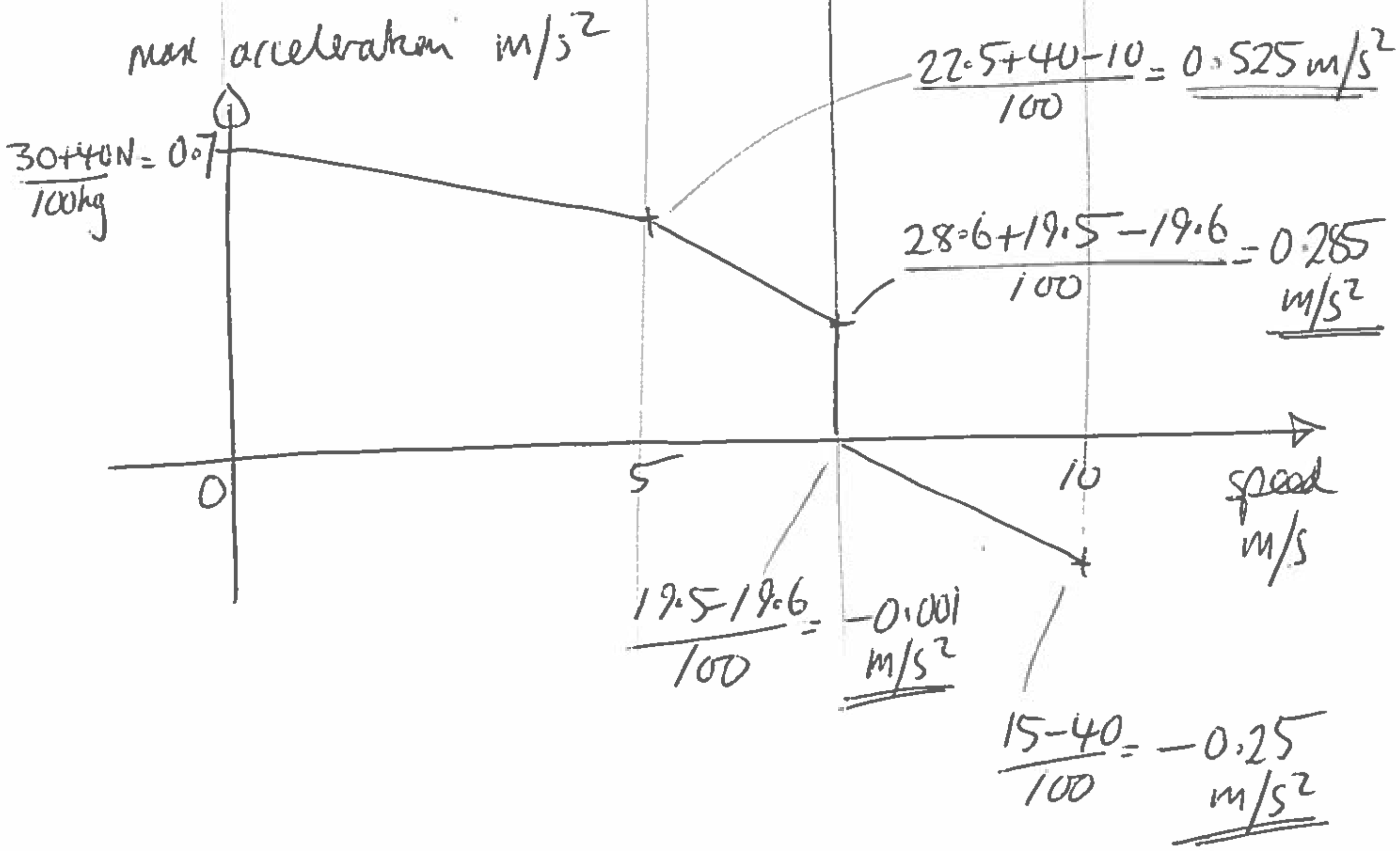
see graphs on next page.

ii) motor on crank instead of front wheel:

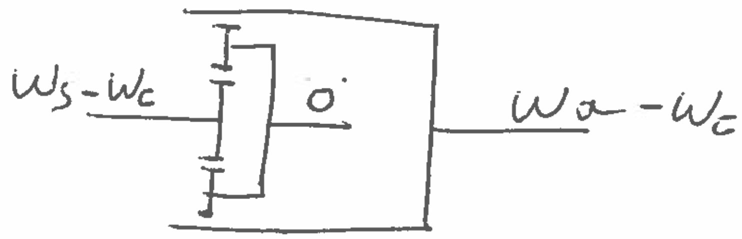
- motor would need to generate larger torque, which generally means more expensive motor
- could take advantage of existing range of speed ratios to provide better matching of motor to load, eg hill climbing
- existing transmission might need strengthening to cope with rider + motor torque.

b) i)


force F/N
motor
200 W
40
30
20
10
0
28·6N
cyclist
22·5N
19·5N
(cyclist)
19·6N (bicycle)
15N
10N
bicycle
40 N
F = 0·4V²
0 1 2 3 4 5 6 7 8 9 10
speed V / m/s


max acceleration m/s²
30+40N / 100kg = 0·7
22·5+40−10 / 100 = 0·525 m/s²
28·6+19·5−19·6 / 100 = 0·285 m/s²
0
5
10
speed m/s
19·5−19·6 / 100 = −0·001 m/s²
15−40 / 100 = −0·25 m/s²

3. a) $\omega_s = (1+R)\omega_c - R\omega_a \qquad R = \frac{A}{S}$.



apply $-\omega_c$ to whole system.

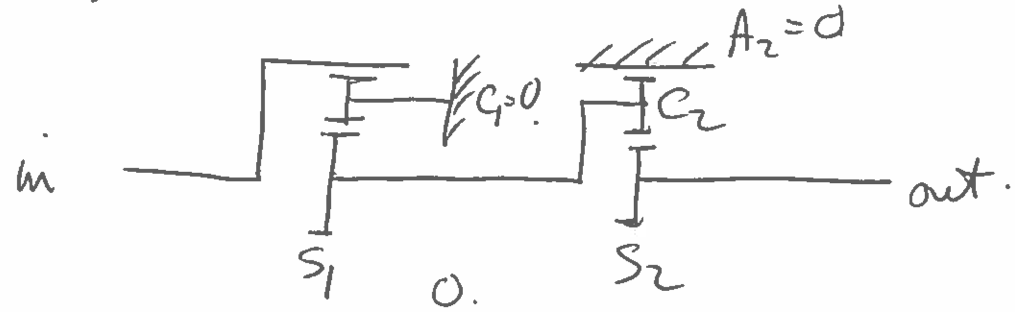
$$\frac{\omega_a - \omega_c}{\omega_s - \omega_c} = -\frac{1}{R}$$

$$R(\omega_a - \omega_c) = -(\omega_s - \omega_c)$$

$$R(\omega_a - \omega_c) - \omega_c = -\omega_s$$

$$-R\omega_a + \omega_c(1+R) = \omega_s$$

b) i) A_2 forced



$$\omega_{s_1} = \cancelto{0}{(1+R_1)\omega_{c_1}} - R_1\omega_{A_1} \qquad \omega_{s_2} = (1+R)\omega_{c_2} - R_2\cancelto{0}{\omega_{A_2}}$$

$$\omega_{s_1} = -R_1\omega_{A_1} \qquad \omega_{c_2} = \frac{\omega_{s_2}}{1+R_2}$$

but $\omega_{s_1} = \omega_{c_2}$

$$\therefore \quad -R_1\omega_{A_1} = \frac{\omega_{s_2}}{1+R_2}$$

$$\frac{\omega_{A_1}}{\omega_{s_2}} = -\frac{1}{R_1(1+R_2)} = -\frac{1}{4(1+3)} = -\frac{1}{16}$$

ii) 300 rpm, 2MW input

$$\omega_{in} = 300 \cdot \frac{2\pi}{60} = 10\pi \text{ rad/s}$$

$$P_{in} = \omega_{in} T_{in}$$

$$\therefore T_{in} = \frac{P_{in}}{\omega_{in}} = \frac{2 \cdot 10^6}{10\pi} = \frac{2 \cdot 10^5}{\pi} \text{ Nm}$$

find torque on A_2.

first find torque on S_1,

epicyclic 1 : $T_{A_1}\omega_{A_1} + T_{C_1}\omega_{C_1} + T_{S_1}\omega_{S_1} = 0$

let $\omega_{C_1} = 0$ $\quad T_{A_1}\omega_{A_1} = -T_{S_1}\omega_{S_1}$

$$T_{S_1} = -T_{A_1}\frac{\omega_{A_1}}{\omega_{S_1}}$$

now use speed rule to find speed ratio

$$\omega_{S_1} = \cancel{(1+R_1)\omega_{C_1}}^{0} - R_1\omega_{A_1}$$

$$\frac{\omega_{A_1}}{\omega_{S_1}} = -\frac{1}{R}$$

thus $\quad T_{S_1} = -T_{A_1} \cdot \frac{-1}{R} = +\frac{T_{A_1}}{4}$

hence $\quad T_{C_2} = -\frac{T_{A_1}}{4}$ (equal & opposite)

next consider epicyclic 2 to find T_{A_2}:

$$T_{A_2}\omega_{A_2} + T_{C_2}\omega_{C_2} + T_{S_2}\omega_{S_2} = 0$$

let $\omega_{S_2} = 0$ $\quad T_{A_2}\omega_{A_2} = -T_{C_2}\omega_{C_2}$

$$T_{A_2} = -T_{C_2}\frac{\omega_{C_2}}{\omega_{A_2}}$$

use speed rule to find speed ratio

$$\cancel{\omega_{S2}}^{\,0} = (1+R_2)\,\omega_{C2} - R_2\,\omega_{A2}$$

$$\frac{\omega_{C2}}{\omega_{A2}} = + \frac{R_2}{1+R_2}$$

thus $\quad T_{A2} = \dfrac{R_2}{1+R_2}\,\dfrac{T_{A1}}{4} = \dfrac{3}{16}\,T_{A1}$

now $\quad T_{A1} = T_{IN} = \dfrac{2.10^5}{\pi}\ Nm.$

hence $\quad \underline{\underline{T_{A2} = \dfrac{3}{16}\cdot\dfrac{2.10^5}{\pi}\ Nm}}$

c) $\quad A_2$ coupled to A_1

$$\omega_{A1} = \omega_{A2} = \omega_{IN}$$

$$\omega_{S1} = \omega_{C2}$$

$$\omega_{S2} = \omega_{OUT}.$$

$$\omega_{C1} = 0.$$

epicyclic 1 :

$$\omega_{S1} = \cancel{(1+R_1)\,\omega_{C1}}^{\,0.} - R_1\,\omega_{A1}$$

$$\omega_{S1} = -R_1\,\omega_{A1} = -R_1\,\omega_{IN} = \omega_{C2}.$$

epicyclic 2 :

$$\omega_{OUT} = \omega_{S2} = (1+R_2)\,\omega_{C2} - R_2\,\omega_{A2}$$

$$= -R_1\,\omega_{IN}(1+R_2) - R_2\,\omega_{IN}.$$

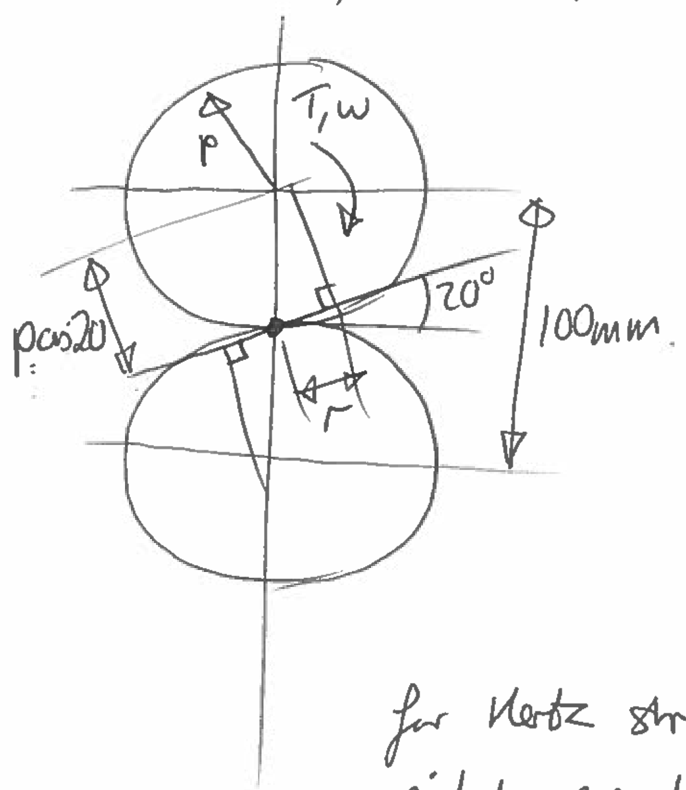
$$\frac{W_{OUT}}{W_{IN}} = -R_1(1+R_2) - R_3$$

$$= -4(1+3) - 3$$

$$= -19$$

$$\frac{W_{IN}}{W_{OUT}} = -\frac{1}{19}$$

4. a) i)



power $P = 50\,kW$

$\omega = 2000\,rpm \cdot \frac{2\pi}{60}$

$= 209.44\ rad/s$

$\therefore T = \frac{P}{\omega} = \underline{\underline{238.7\ Nm}}$

pitch circle radius

$p = 50\,mm$.

$\therefore r = 50 \sin 20$

$r = 17.1\,mm$

for Hertz stress calculation at pitch point $R = \frac{r}{2} = 8.55\,mm$

line contact, peak contact stress p_0

$$p_0 = \left\{ \frac{P' E^*}{\pi R} \right\}^{\frac{1}{2}} \quad \text{(data sheet)}$$

allowable contact force per unit width $P' = \frac{p_0^2 \pi R}{E^*} = \frac{10^8 \cdot \pi \cdot 8.55 \cdot 10^{-3}}{115 \cdot 10^9}$

$P' = \underline{\underline{233.57\ kN/m}}$.

let total contact force be F, such that $F = P'w$ where w is width.

contact force is related to wheel torque T by

$F = \frac{T}{p \cos 20}$ thus. $w = \frac{T}{p \cos 20° P'}$

$$w = \frac{238.7}{0.05 . \cos 20^\circ \; 233.57 . 10^3} = \underline{\underline{21.75 \text{ mm}}}$$

ii) use $\sigma_b = \dfrac{P'_T}{J\,m}$ where $P'_T = P' \cos 20^\circ$

$$P'_T = 233.57 . 10^3 \cos 20^\circ = 219.48 \text{ kN/m}$$

allowable $\sigma_b = 300$ MPa.

imperfectly made gears so assume worst case for J, $J = 0.21$

Thus $\sigma_b = 300 . 10^6 = \dfrac{P'_T}{J m} = \dfrac{219.48 . 10^3}{0.21 . m}$

$\therefore \; m = 3.48$ mm (module must be no less than this)

for integer number of teeth and exactly 100mm centres, choose $m = 4$ mm, giving 25 teeth per wheel.

b) i) radial force on each bearing $P = \frac{F}{2}$

$$P = \frac{1}{2}\frac{T}{p\cos 20} = \frac{1}{2} \cdot \frac{238.7}{0.05\cos 20} = 2540.2\,N.$$

for bearing 6008, dynamic load rating $C = 16{,}800N$

life eqn $L = a_1 a_{23} \left(\frac{C}{P}\right)^p$

for ball bearing exponent $p = 3$

for 2% probability of failure $a_1 = 0.33$

for oil of correct viscosity $a_{23} = 1$

$$\therefore \quad L = 0.33 \, . \, 1 \, . \left(\frac{16\,800}{2540.2}\right)^3$$

$$= 95.43 \, . \, 10^6 \text{ revolutions}$$

$$\text{life in hours} = \frac{95.43 \, . \, 10^6 \text{ revs.}}{2000 \text{ rpm} \, . \, 60 \text{ mins/hour}}$$

$$= 795.3 \text{ hours}$$

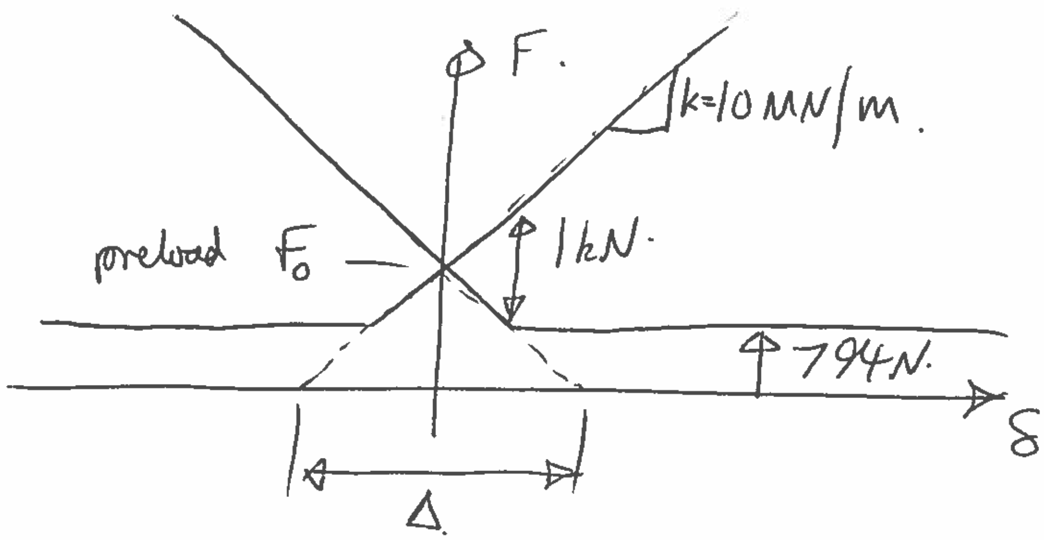
ii) potential problem is unintended axial forces in bearings due to mismatched tolerances or thermal expansion. Solution is to arrange for one fixed bearing and one floating bearing.

c) radial force per bearing $F_r = 2540.2\,N$.

induced axial force $F_a = \dfrac{F_r}{2Y} = \dfrac{2540.2}{2 \cdot 1.6}$

$= \underline{\underline{794\,N}}$

draw axial force-deflection diagram



preload force $F_0 = 794\,N + \dfrac{1000\,N}{2}$

$\underline{\underline{F_0 = 1294\,N}}$

preload displacement $\Delta = 2 \cdot \dfrac{F_0}{k}$

$= \dfrac{2 \cdot 1294}{10 \cdot 10^6}$

$= \underline{\underline{0.256\ mm}}$

ENGINEERING TRIPOS PART IIA 2015

COMMENTS ON QUESTIONS, MODULE 3C8: MACHINE DESIGN

Q1 Contact mechanics

The least popular question. Parts (a) and (b)(i) were answered satisfactorily. Very few candidates obtained the correct answer to part (ii); most solutions didn't demonstrate appreciation of the equilibrium of applied forces F and P and the contact pressure. In part (iii) most candidates were looking for a F proportional to P relationship, and didn't account for the contact width $2b$ varying with P.

Q2 Power matching

Part (a) was generally answered well. Part (b)(i) was more challenging; solutions taking a graphical approach were more successful than those taking an algebraic approach. In part (b)(ii) a wide range of practical design issues were discussed, but few candidates identified the significant power matching issues.

Q3 Epicyclic gear

The proof of the epicyclic speed rule was found to be difficult, many answers unnecessarily involved torques, powers or tabular methods. The calculation of the restraint torque in part (b)(ii) was also challenging; a common mistake was to omit the torque of the grounded carrier C1 when considering torque equilibrium.

Q4 Spur gears and bearings

This question comprised five short parts. Units on intermediate numerical values were often absent or wrong, which probably contributed to a general lack of accuracy in calculation. There was often confusion about how torque on the shafts related to contact force along the pressure line, bending force on the teeth, and radial force on the bearings.

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