3D4 2017 Solutions

Qu 1. (a) (i) Torque at C must be T_0

(ii) At B twist to left in AB
= twist to right in BC
∴ twist = 0.

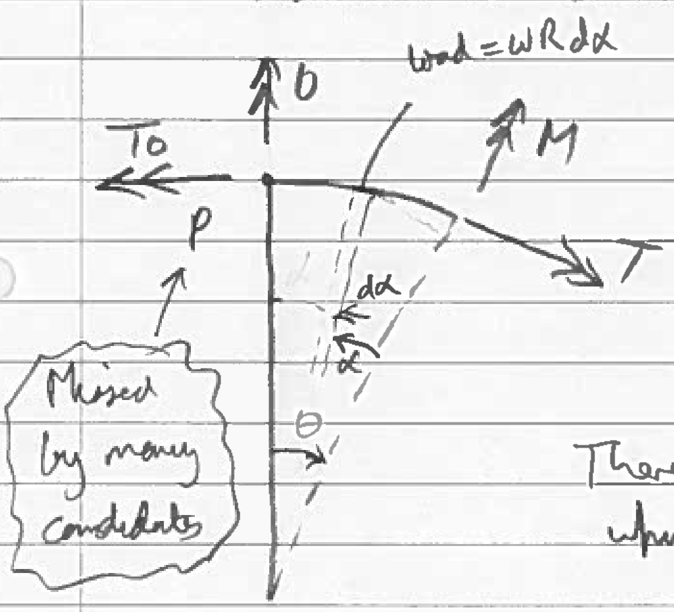
There must be no rotation in bending direction at B. (i.e. it must be horizontal.

(b) T_0 is unknown. Problem is once statically indeterminate.

Rotation is zero at A & B.

∴ Integrate rotation from A to B, which will be function of T_0. Setting rotation = 0 $\Rightarrow T_0$

No need to consider moment yet.



Find torque due to small element of load

$$dT = wR.d\alpha \; R(1-\cos\alpha)$$

There will be a vertical reaction upwards at A of $P = wR\frac{\pi}{6}$

Torsional equilibrium $T = T_0 \cos\theta + PR(1-\cos\theta)$

$$- wR^2\int_0^\theta (1-\cos\alpha)\,d\alpha$$

Many omitted this term, assuming load was lumped at midpoint

$$\int_0^\theta (1-\cos\alpha)\,d\alpha = \theta - \left[\sin\alpha\right]_0^\theta$$

$$= \theta - \sin\theta$$

$$\therefore T = T_0\cos\theta + \frac{wR^2\pi}{6}(1-\cos\theta) - wR^2\theta + wR^2\sin\theta$$

Rate of twist $= \dfrac{T}{GJ}$.

$\therefore$ Total twist $= GJ.\displaystyle\int_0^{\pi/6} T\,d\theta = 0$ to satisfy B.C.

$$\therefore 0 = T_0\underbrace{\int_0^{\pi/6}\cos\theta}_{=\sin\pi/6 = 1/2} + \frac{wR^2\pi}{6}\cdot\frac{\pi}{6} - \frac{wR^2\pi}{6}\underbrace{\int_0^{\pi/6}\cos\theta}_{=1/2} - \frac{wR^2}{2}\frac{\pi^2}{36}$$

$$+ wR^2\underbrace{\int_0^{\pi/6}\sin\theta}_{(1-\frac{\sqrt{3}}{2})}$$

$$\Rightarrow \frac{T_0}{2} + wR^2\underbrace{\left(\cancel{\frac{\pi^2}{36}} - \frac{\pi}{12} - \cancel{\frac{\pi^2}{72}}^{\pi^2/72} + 1 - \frac{\sqrt{3}}{2}\right)}_{+0.00925} = 0$$

$$\Rightarrow \underline{\underline{T_0 = -0.0185\,wR^2}}$$

At $\theta = \dfrac{\pi}{6}$ $\quad$ $T = \underline{\underline{+0.0305\,wR^2}}$

To calculate M at θ

$$M = T_0 \sin\theta - P R \sin\theta + \underbrace{wR\int_0^{\theta} d\alpha \, . R \sin\alpha}_{= wR^2 (1 - \cos\theta)}$$

at $\theta = \pi/6$ $\qquad \dfrac{M}{wR^2} = -0.0185 - \dfrac{\pi}{6} \cdot \dfrac{1}{2} + 1 - \dfrac{\sqrt{3}}{2}$

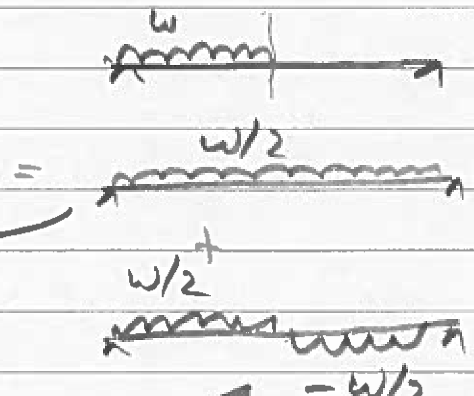
$$= -0.137$$

(c) Two possible answers (i) Symmetry no longer applies

∴ We can't apply any conditions at B and the torque in the two halves at A & C will be different and the support reactions will also be different.

∴ Treat each half in isolation and integrate torque and rotation towards B and use equality of slope, rotation and deflection to solve for the unknown reactions.

OR (ii) Split into



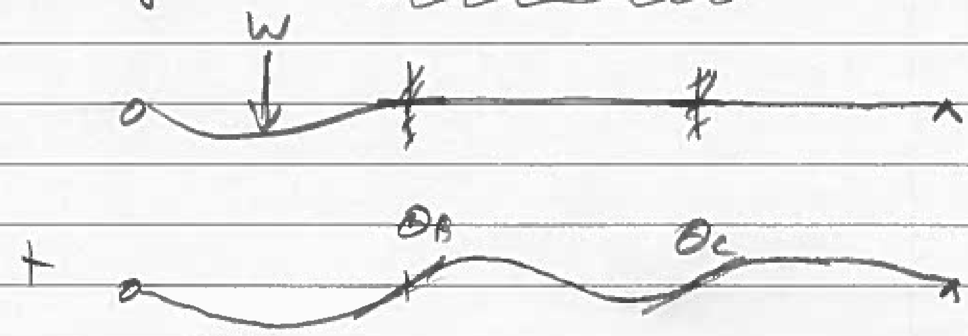
half of result in (b).

symmetry here says rotation & deflection both zero

∴ M = T = 0

2 (a) Stiffness analysis {Follows Lecture Notes}

Split into clamped pieces at B & C

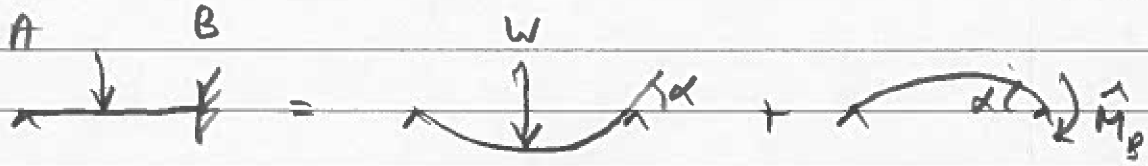


Variables will be joint rotations

$$\begin{bmatrix} K \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} \hat{M}_B \\ \hat{M}_C \end{bmatrix}$$

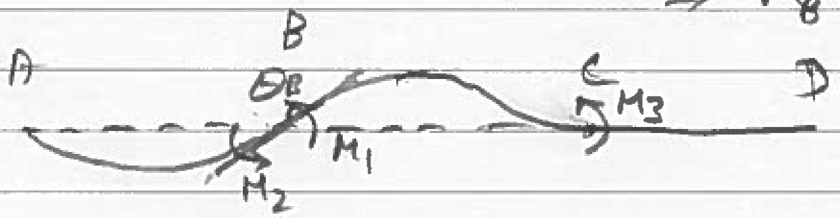
↑ out of balance moments due to load.

Use data book coefficients to work out various terms



$$\alpha = \frac{WL^2}{16\cancel{EI}} = \frac{\hat{M}_B L}{3\cancel{EI}}$$

$$\Rightarrow \hat{M}_B = \frac{3WL}{16}$$



$$M_2 = \frac{3EI\theta_B}{L} \qquad M_1 = \frac{4EI}{L}\theta_B \qquad M_3 = \frac{2EI}{L}\theta_B$$

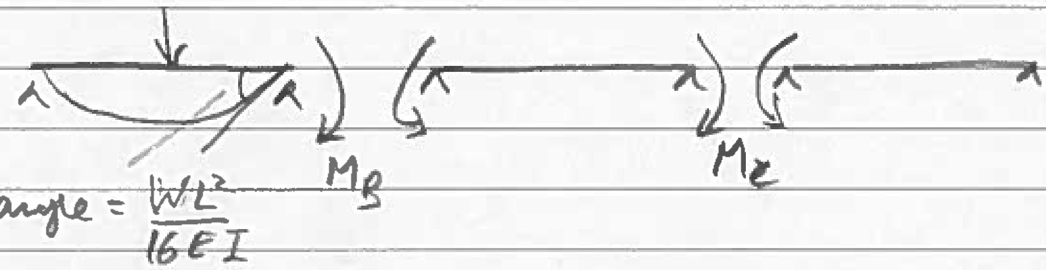
add to get M_B

Similarly for C

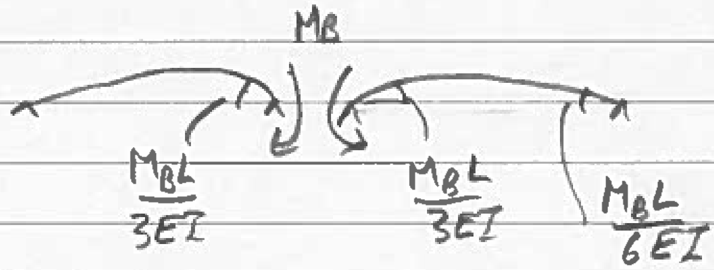
$$\Rightarrow \frac{EI}{L}\underbrace{\begin{bmatrix}7 & 2\\ 2 & 7\end{bmatrix}}_{\text{Stiffness matrix}}\begin{bmatrix}\theta_B\\ \theta_C\end{bmatrix} = \frac{3L}{16}\begin{bmatrix}W\\ 0\end{bmatrix}$$

(b) Flexibility analysis

Split structure into determinate elements



Consider effect of M_B



Similarly for M_C

$\therefore$ At B angular change $\frac{2M_BL}{3EI} + \frac{M_CL}{6EI} - \frac{WL^2}{16EI} = 0$

Similarly for C

Leads to

$$\underbrace{\frac{L}{6\cancel{EI}}\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}}_{\text{Flexibility Matrix}}\begin{bmatrix}M_B\\ M_C\end{bmatrix} = \frac{WL^2}{16\cancel{EI}}\begin{bmatrix}1\\ 0\end{bmatrix}$$

(c)

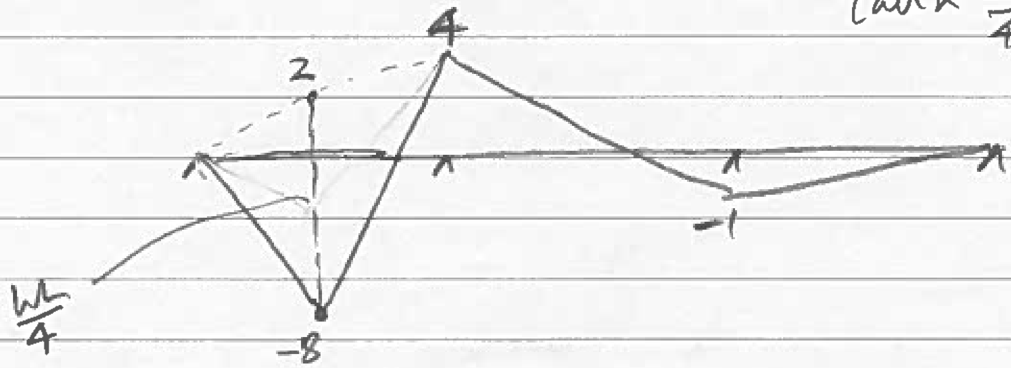
$$\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}\begin{bmatrix}M_B\\ M_C\end{bmatrix} = \frac{3WL}{8}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\begin{bmatrix}M_B\\ M_C\end{bmatrix} = \frac{3WL}{\underbrace{15}_{\text{Det}}\cdot 8}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= \frac{WL}{40}\begin{bmatrix}4\\ -1\end{bmatrix}$$

∴ B.M diag

(all × $\frac{WL}{40}$)



Very few got a diagram anything like this

Qu3. (a)

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} s+4 & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$

Critical when det $=0$ $\Rightarrow s^2+4s-s^2c^2=0$

$\Rightarrow s\left(s(1-c^2)+4\right)=0$ so $s=0$
or $s(1-c^2)+4=0$

P/P_E	s	c	$s(1-c^2)$
1·4	1·6782	1·6557	−2·922
1·6	1·2240	2·4348	−6·032

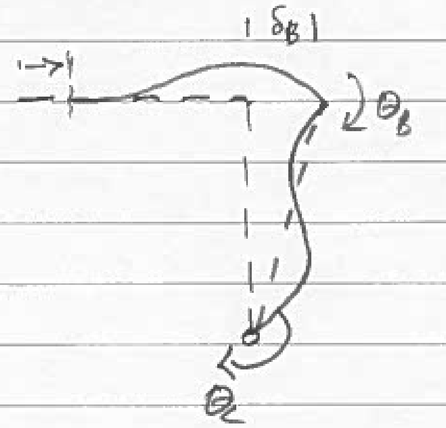
Linearly interpolate $\dfrac{4-2{\cdot}922}{6{\cdot}032-2{\cdot}922} = 0{\cdot}35$

$\therefore P/P_E = 1{\cdot}4 + 0{\cdot}35 \times 0{\cdot}2 = \underline{\underline{1{\cdot}47}}$

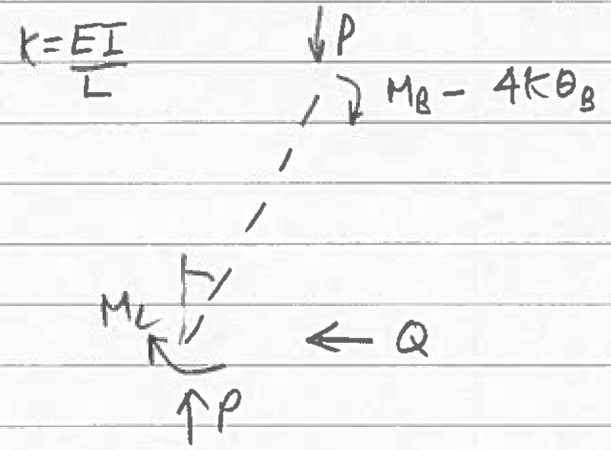
or by plotting

$\therefore P = \dfrac{\pi^2 EI}{L^2} \times 1{\cdot}47 = 14{\cdot}5\,\dfrac{EI}{L^2}$

(b) Displacements



Focus on BC



For BC original equations are valid if we replace θ_B by $\theta_B - \frac{\delta_B}{L}$ and θ_C by $\theta_C - \frac{\delta_C}{L}$

So

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_B - \delta_B/L \\ \theta_C - \delta_B/L \end{bmatrix} + \underbrace{\frac{EI}{L} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{for } AB} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

$$= \frac{EI}{L} \begin{bmatrix} s+4 & sc & -(s+sc)/L \\ sc & s & -(s+sc)/L \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \delta_B \end{bmatrix}$$

Need a 3rd equation - find Q

$$QL + M_B - 4K\theta_B + M_C + P\delta_B = 0 \qquad -①$$

$$\frac{M_B}{K} = (s+4)\,\theta_B + sc\,\theta_C - \frac{(s+sc)}{L}\,\delta_B$$

$$\frac{M_C}{K} = sc\,\theta_B + s\,\theta_C - \frac{(s+sc)}{L}\,\delta_C$$

Substitute into ① gives

$$Q = -\frac{K}{L}(s+sc)\,\theta_B - \frac{K}{L}(s+sc)\,\theta_C + \left(\frac{2(s+sc)}{L^2} - \frac{P}{L}\right)\delta_B$$

So

$$\begin{bmatrix} M_B \\ M_C \\ Q \end{bmatrix} = K \begin{bmatrix} s+4 & sc & -(s+sc)/L \\ sc & s & -(s+sc)/L \\ -(s+sc)/L & -(s+sc)/L & \frac{2(s+sc)}{L^2} - \frac{P}{L} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \delta_B \end{bmatrix}$$

Q4 $\Pi = \frac{1}{2}k \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$$- \frac{P}{2L} \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Write as $\pi = \frac{1}{2} w^T \underline{K_T} \, \underline{w}$

where $K_T = \begin{bmatrix} k - 2P/L & P/L \\ P/L & k - 2P/L \end{bmatrix} \equiv \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

Eigenvalues $\begin{vmatrix} a - \lambda & b \\ b & a - \lambda \end{vmatrix} = 0$

leads to $a - \lambda = \pm b$

$\lambda_1 = a + b = k - P/L \Rightarrow P = kL$ (First c.L)

$\lambda_2 = a - b = k - 3P/L \quad P = 3kL$ (second c.L)

1st critical load $P = kL$ $\quad K_T = \begin{bmatrix} k & k \\ k & k \end{bmatrix}$ eigenvalue $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2nd critical load $\quad K_T = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

N.B. lowest buckling load is skew-symmetric

(b)

(i) $$\Pi = \frac{1}{2}k\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$-\frac{P}{2L}\begin{bmatrix} w_1-\varepsilon L & w_2-\varepsilon L \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} w_1-\varepsilon L \\ w_2-\varepsilon L \end{bmatrix}$$

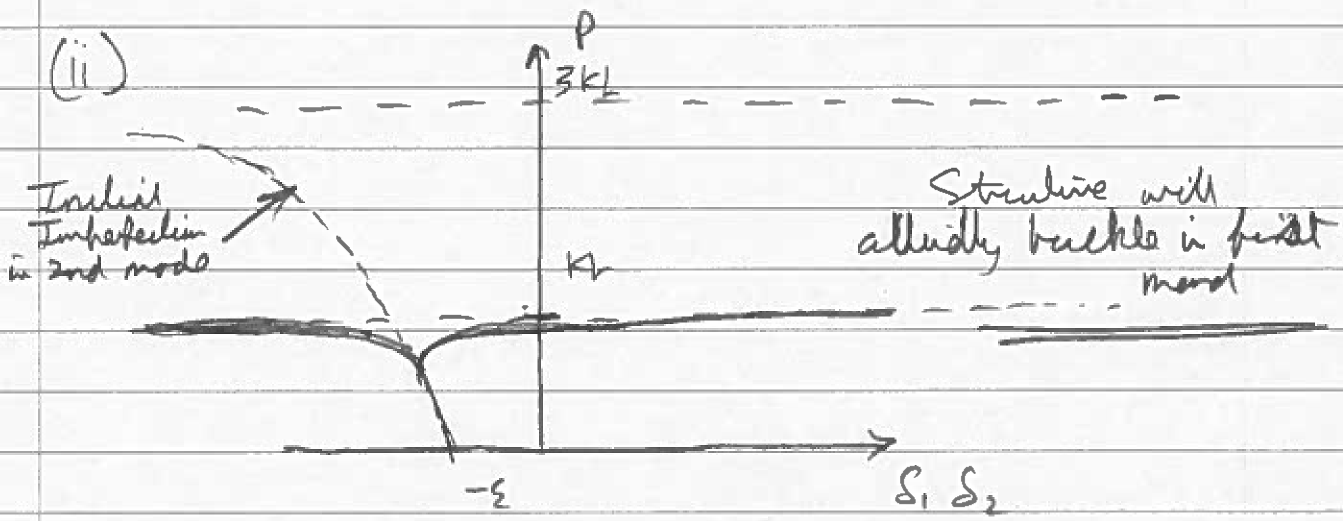
$$= \frac{1}{2}\varepsilon\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$-\frac{P}{2L}\begin{bmatrix} w_1-\varepsilon L & w_2-\varepsilon L \end{bmatrix}\begin{bmatrix} 2(w_1-\varepsilon L) & -1.(w_2-\varepsilon L) \\ -(w_1-\varepsilon L) & 2(w_2-\varepsilon L) \end{bmatrix}$$

$$= \frac{1}{2}k\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \frac{P}{2L}\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$+ \begin{bmatrix} P\varepsilon & P\varepsilon \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

(ii)



3D4 Comments for crib

1. Twisted bridge.

Quite a few had no idea where to start on this question. For those who could, the biggest problem was the omission of the support reaction and the failure to account for the position of the load on the bridge. All assumed that the resultant of the weight acted at the mid-point of the part of the bridge they were considering. This avoided an integration but makes quite a big difference to the answer. The final part was done quite well; many recognised the symmetry and skew-symmetry parts, but failed to recognise that the skew symmetry meant that the moment and torque must be zero at the centre.

2. Difference between Stiffness and Flexibility Methods.

Very disappointing. I had given them an example in their notes this year of virtually the same problem solved by both methods and this question followed the lecture notes closely, with just a different loading case. However, most had not read their notes and even fewer had any understanding. Many got the two methods the wrong way round. Some tried to use Macaulay for one or both of the problems. The last part was intended to be a trivial solution of a 2×2 matrix they had already derived but few did it that way and even fewer could calculate the moment under the point load. There were many examples of completely nonsensical bending moment diagrams; not just wrong calculation errors but utter drivel.

3. s and c functions for buckling of frames.

The no-sway frame was fairly straightforward and most were correct. The sway frame was more complex with three types of attempts; those who had no idea at all, those who got most of it correct and explained what they were doing, and those who were clearly reproducing their lecture notes with no explanation, simply writing down solutions as though they could do a page of algebra in their heads.

4. Buckling of 3-segment bar.

They were asked to use the potential energy function to determine the buckling loads and modes. Most could derive the PE expression but some had trouble taking account of the geometry constraint relating the angles of the three elements to the two primary variables. Those who didn't write the PE function in matrix notation had trouble determining the buckling loads, and many of those who did couldn't find the eigenvalues. Those who tackled the later parts did them reasonably well, but there were many trivial mistakes and few picked up on the fact that despite the initial imperfections being in one of the buckling modes, the structure would actually buckle in the other mode.