Q.1.(a)

Horizontal centroid by inspection: $5.5t$

Vertical centroid by inspection: $4t$

Second moment of area, I_{xx}, determined from built-up sections using parallel axis theorem:

$$
\begin{aligned}
I'_{xx} &= \sum I_{xx} + \left(A_s \cdot y_s^2\right) \\
&= \left[\left(\frac{6t \cdot t^3}{12}\right) + \left(6t \cdot t \cdot \left(-4t + \frac{t}{2}\right)^2\right)\right] + \left[\left(\frac{(6t)^3 \cdot t}{12}\right) + \left(6t \cdot t \cdot 0^2\right)\right] + \left[\left(\frac{6t \cdot t^3}{12}\right) + \left(6t \cdot t \cdot \left(4t - \frac{t}{2}\right)^2\right)\right] \\
&= \left[\left(\frac{t^4}{2}\right) + \left(\frac{147t^4}{2}\right)\right] + \left[\left(\frac{36t^4}{2}\right) + 0\right] + \left[\left(\frac{t^4}{2}\right) + \left(\frac{147t^4}{2}\right)\right] \\
&= 166t^4
\end{aligned}
$$

Second moment of area, I_{yy}, determined from built-up sections using parallel axis theorem:

$$
\begin{aligned}
I'_{yy} &= \sum I_{yy} + \left(A_s \cdot x_s^2\right) \\
&= \left[\left(\frac{(6t)^3 \cdot t}{12}\right) + \left(6t \cdot t \cdot \left(-\frac{6t}{2} + \frac{t}{2}\right)^2\right)\right] + \left[\left(\frac{6t \cdot t^3}{12}\right) + \left(6t \cdot t \cdot 0^2\right)\right] + \left[\left(\frac{(6t)^3 \cdot t}{12}\right) + \left(6t \cdot t \cdot \left(\frac{6t}{2} - \frac{t}{2}\right)^2\right)\right] \\
&= \left[\left(\frac{36t^4}{2}\right) + \left(\frac{75t^4}{2}\right)\right] + \left[\left(\frac{t^4}{2}\right) + 0\right] + \left[\left(\frac{36t^4}{2}\right) + \left(\frac{75t^4}{2}\right)\right] \\
&= 111.5t^4
\end{aligned}
$$

Second moment of area, I_{xy}, determined from built-up sections using parallel axis theorem:

$$
\begin{aligned}
I'_{xy} &= \sum I_{xy} + \left(A_s \cdot x_s \cdot y_s\right) \\
&= \left[0 + \left(6t \cdot t \cdot \left(-\frac{6t}{2} + \frac{t}{2}\right) \cdot \left(\frac{8t}{2} - \frac{t}{2}\right)\right)\right] + \left[0 + (6t \cdot t \cdot 0 \cdot 0)\right] + \left[0 + \left(6t \cdot t \cdot \left(\frac{6t}{2} - \frac{t}{2}\right) \cdot \left(\frac{-8t}{2} + \frac{t}{2}\right)\right)\right] \\
&= \left[\frac{-105t^4}{2}\right] + \left[0\right] + \left[\frac{-105t^4}{2}\right] \\
&= -105t^4
\end{aligned}
$$

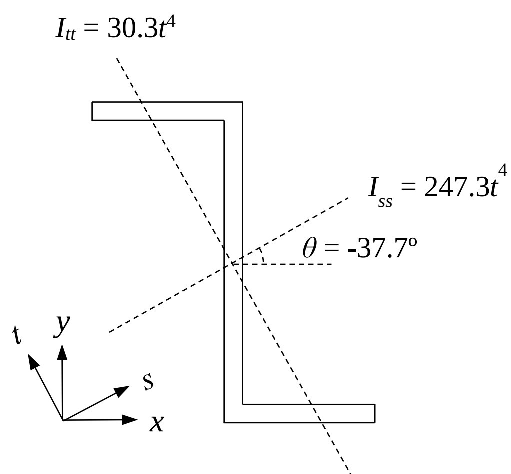
Q.1.(b)

Principal second moment of area, I_{ss}, found where $I_{st} = 0$ using Mohr's circle geometry:

$$
\begin{aligned}
I_{ss} &= \frac{1}{2}\left(I_{xx} + I_{yy}\right) + \sqrt{I_{xy}^2 + \left(I_{xx} - \left(\frac{1}{2}\left(I_{xx} + I_{yy}\right)\right)\right)^2} \\
&= \frac{1}{2}\left(166t^4 + 111.5t^4\right) + \sqrt{\left(-105t^4\right)^2 + \left(166t^4 - \left(\frac{1}{2}\left(166t^4 + 111.5t^4\right)\right)\right)^2} \\
&= 138.8t^4 + 108.5t^4 \\
&= 247.3t^4
\end{aligned}
$$

Principal second moment of area, I_{tt}, found where $I_{st} = 0$ using Mohr's circle geometry:

$$
\begin{aligned}
I_{tt} &= \frac{1}{2}\left(I_{xx} + I_{yy}\right) - \sqrt{I_{xy}^2 + \left(I_{xx} - \left(\frac{1}{2}\left(I_{xx} + I_{yy}\right)\right)\right)^2} \\
&= \frac{1}{2}\left(166t^4 + 111.5t^4\right) + \sqrt{\left(-105t^4\right)^2 + \left(166t^4 - \left(\frac{1}{2}\left(166t^4 + 111.5t^4\right)\right)\right)^2} \\
&= 138.8t^4 - 108.5t^4 \\
&= 30.3t^4
\end{aligned}
$$



Q.1.(c)

Orientation of principal second moments of area:

$$
\begin{aligned}
tan\left(2\theta\right) &= \frac{I_{xy}}{I_{xx} - \frac{1}{2}\left(I_{xx} + I_{yy}\right)} \\
&= \frac{-105t^4}{166t^4 - 138.8t^4} \\
&= -3.86 \\
\therefore \theta &= -37.7^{\circ}
\end{aligned}
$$

From the structures data book, the end deflection of a beam of length L when subjected to uniformly distributed load (i.e. self-weight) is $\frac{wL^4}{8EI}$:

$$\delta_s = \frac{sin\theta \cdot wL^4}{8EI_{tt}} = \frac{sin\theta \cdot wL^4}{8E \cdot 30.3t^4}$$
$$= 2.5 \times 10^{-3} \cdot \frac{wL^4}{Et^4}$$
$$\delta_t = \frac{cos\theta \cdot wL^4}{8EI_{ss}} = \frac{cos\theta \cdot wL^4}{8E \cdot 247.3t^4}$$
$$= 4.0 \times 10^{-4} \cdot \frac{wL^4}{Et^4}$$
$$\delta = \sqrt{\delta_s^2 + \delta_t^2}$$
$$= 2.5 \times 10^{-3} \cdot \frac{wL^4}{Et^4}$$

Q.1.(d) St. Venant's torsion constant:

$$J = \sum \frac{1}{3} \cdot b \cdot t^3 = \left[\frac{1}{3} \cdot 6t \cdot t^3\right] + \left[\frac{1}{3} \cdot 6t \cdot t^3\right] + \left[\frac{1}{3} \cdot 6t \cdot t^3\right]$$
$$= 6t^4$$

Q.1.(e) Rotation of the tip at the centroid for unrestrained warping:

$$\theta_u = \frac{TL}{GJ} = \frac{M \cdot 100t}{G \cdot 6t^4}$$
$$= 16.7\frac{M}{Gt^3}$$

Restrained warping constant, with d taken as vertical distance between flange centroids ($7t$):

$$\Gamma = \frac{d^2}{4}I_{yy} = \frac{(7t)^2}{4}111.5t^4$$
$$= 1366t^6$$

Characteristic length:

$$\lambda = \sqrt{\frac{E\Gamma}{GJ}} = \sqrt{\frac{E \cdot 1366t^6}{G \cdot 6t^4}} = 15.1 \cdot \sqrt{2\left(1+\nu\right)} \cdot t = 15.1 \cdot \sqrt{2.6} \cdot t$$
$$= 24.3t$$

Effective length:

$$L^{'} \approx L - \lambda = 100t - 24.3t = 75.7t$$

Rotation of the tip at the centroid for restrained warping:

$$\theta_r = \frac{TL}{GJ} = \frac{M \cdot 75.7t}{G \cdot 6t^4} = 12.6\frac{M}{Gt^3}$$

Reduction in twist at the tip:

$$1 - \frac{\theta_r}{\theta_u} = 1 - \frac{12.6\frac{M}{Gt^3}}{16.7\frac{M}{Gt^3}} = 0.25$$

Therefore restrained warping results in a 25% reduction in twist at the tip.

Q.2.(a)

Using Macaulay's method:

$$-EI\frac{d^2v}{dx^2} = R_A x + R_B\{x-3\} - 6\{x-6\} - 6\{x-9\}$$
$$\therefore -EIv = R_A\frac{x^3}{6} + R_B\frac{\{x-3\}^3}{6} - \frac{6\{x-6\}^3}{6} - \frac{6\{x-9\}^3}{6} + Ax + B$$

Applying boundary conditions @ $x = 0 \rightarrow$ v = 0 $\therefore B = 0$

Applying boundary conditions @ $x = 3 \rightarrow$ v = 0:

$$\begin{aligned} 0 &= R_A\frac{x^3}{6} + R_B\frac{\{x-3\}^3}{6} - \frac{6\{x-6\}^3}{6} - \frac{6\{x-9\}^3}{6} + Ax \\ &= R_A\frac{3^3}{6} + R_B\frac{\{3-3\}^3}{6} - \frac{6\{3-6\}^3}{6} - \frac{6\{3-9\}^3}{6} + 3A \\ &= 4.5R_A + 3A \therefore A = -1.5R_A \end{aligned}$$

Take moments about C:

$$12R_A + 9R_B = (6\cdot 6) + (6\cdot 3)$$
$$12R_A + 9R_B = 54 \therefore R_A = 4.5 - 0.75R_B$$

Applying boundary conditions @ $x = 12 \rightarrow$ v = 0 given A = -1.5R$_A$ and $R_A = 4.5 - 0.75R_B$:

$$\begin{aligned} 0 &= R_A\frac{x^3}{6} + R_B\frac{\{x-3\}^3}{6} - \frac{6\{x-6\}^3}{6} - \frac{6\{x-9\}^3}{6} + Ax \\ &= R_A\frac{12^3}{6} + R_B\frac{\{12-3\}^3}{6} - \frac{6\{12-6\}^3}{6} - \frac{6\{12-9\}^3}{6} + 12A \\ &= 288R_A + 121.5R_B - 243 + 12A \\ &= 288\,(4.5 - 0.75R_B) + 121.5R_B - 243 - 18R_A \\ &= 1215 - 202.5R_B + 121.5R_B - 243 \therefore R_B = 12\text{kN} \end{aligned}$$

Determine remaining reaction forces:

$$\begin{aligned} R_A &= 4.5 - 0.75R_B \\ &= 4.5 - 0.75\cdot 12 = -4.5\text{kN} \end{aligned}$$

Finally, from vertical equilibrium:

$$\begin{aligned} R_C &= 6 + 6 - R_B - R_A \\ &= 6 + 6 - 12 + 4.5 = 4.5\text{kN} \end{aligned}$$

Q.2.(b)

From part (a):

$$\begin{aligned}
-EIv &= R_A\frac{x^3}{6} + R_B\frac{\{x-3\}^3}{6} - 6\frac{\{x-6\}^3}{6} - 6\frac{\{x-9\}^3}{6} + Ax \\
&= -4.5\frac{x^3}{6} + 12\frac{\{x-3\}^3}{6} - \{x-6\}^3 - \{x-9\}^3 + 6.75x \\
\therefore v &= \frac{1}{EI}\left(0.75x^3 - 2\{x-3\}^3 + \{x-6\}^3 + \{x-9\}^3 - 6.75x\right)
\end{aligned}$$

Q.2.(c)

Maximum deflection where $\frac{dv}{dx} = 0$:

$$\frac{dv}{dx} = \frac{1}{EI}\left(2.25x^2 - 6\{x-3\}^2 + 3\{x-6\}^2 + 3\{x-9\}^2 - 6.75\right)$$

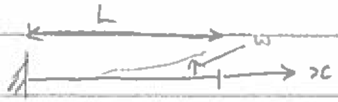
Maximum deflection expected at $6 < x < 9$, therefore expand the first two Macaulay brackets:

$$\begin{aligned}
0 &= \frac{1}{EI}\left(2.25x^2 - 6\{x-3\}^2 + 3\{x-6\}^2 + 3\{x-9\}^2 - 6.75\right) \\
&= \frac{1}{EI}\left(2.25x^2 - 6x^2 + 18x + 18x - 54 + 3x^2 - 18x - 18x + 108 - 6.75\right) \\
&= \frac{1}{EI}\left(-0.75x^2 + 47.25\right) \therefore x = 7.94 \text{ m}
\end{aligned}$$

Find deflection at x:

$$\begin{aligned}
v &= \frac{1}{EI}\left(0.75x^3 - 2\{x-3\}^3 + \{x-6\}^3 + \{x-9\}^3 - 6.75x\right) \\
&= \frac{1}{2\times 10^4}\left(375.4 - 241.1 + 7.3 + 0 - 53.6\right) \\
&= 4.4 \times 10^3 \text{ m} = 4.4 \text{ mm}
\end{aligned}$$

3D4, 2019 [Q3 p1]

Q3.  $w = L \sum_{j=2}^{N} \alpha_j \left(\frac{x}{L}\right)^j$

a) Start at $j=2$ because deflection and slope are zero at $x=0$. [10%]
(i.e. $\alpha_0 = 0, \alpha_1 = 0$)

b) INT. STRAIN ENERGY $= \frac{1}{2}\int_0^L EI (w'')^2 dx.$ $\qquad w'' = L \sum_{j=2}^{N} j(j-1)\alpha_j \frac{x^{j-2}}{L^j}$

let $s = x/L$; $\quad dx = L\,ds$ $\qquad = \frac{1}{L}\sum_{j=2} j(j-1)\,\alpha_j s^{j-2}$

$$\text{INT. STRAIN ENERGY} = \frac{L}{2L^2} EI \int_0^1 \sum_{i=2}^{N}\sum_{j=2}^{N} i(i-1)j(j-1)\alpha_i\alpha_j \underbrace{s^{i-2}s^{j-2}}_{=s^{i+j-4}} ds$$

and $\int_0^1 s^{i+j-4} ds = \frac{1}{i+j-3}$

$$\text{so INT. STRAIN ENERGY} = \frac{1}{2L} EI \sum_{i=2}^{N}\sum_{j=2}^{N} \frac{i\,j\,(i-1)(j-1)}{i+j-3}\alpha_i\alpha_j$$

$$\text{WD by EXT LOADS} = P\int_0^L \frac{1}{2}(w')^2 dx = \frac{PL}{2}\int_0^1 (w')^2 ds$$

$$w' = \sum L j \alpha_j \frac{x^{j-1}}{L^j} = \sum j\alpha_j \frac{x^{j-1}}{L^{j-1}} = \sum j\alpha_j s^{j-1}$$

$$\int_0^1 (w')^2 ds = \int_0^1 \sum_i \sum_j ij\alpha_i\alpha_j \underbrace{s^{i-1}s^{j-1}}_{s^{i+j-2}} ds$$

$$\int_0^1 s^{i+j-2} ds = \frac{1}{i+j-1}$$

$$\text{WD by EXT LOADS} = \frac{PL}{2}\sum_i\sum_j \left(\frac{i\,j}{i+j-1}\right)\alpha_i\alpha_j$$

$$\text{TOT. P.E.} = \text{INT STRAIN EN.} - \text{WD by EXT LOADS} = \frac{1}{2}\sum_{i=2}^{N}\sum_{j=2}^{N}\left[\frac{EI}{L}\frac{ij(i-1)(j-1)}{i+j-3} - PL\frac{ij}{i+j-1}\right]\alpha_i\alpha_j$$

$$\text{i.e. } K_{ij} = \frac{EI}{L}\frac{ij(i-1)(j-1)}{i+j-3} - PL\frac{ij}{i+j-1}$$ [50%]

Q3 cont'd.

c) Let $N=2 \rightarrow i=2,\ j=2$

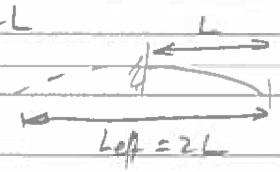
$$\therefore k_{22} = \frac{EI}{L}\left(\frac{2.2.1.1.}{4-3}\right) - PL\left[\frac{2.2.}{4-1}\right]$$

$$= \frac{4EI}{L} - \frac{4PL}{3}$$

P_{crit} when $k_{22}=0 \Rightarrow P_{cr} \simeq \frac{3EI}{L^2}$ estimate

[10%]

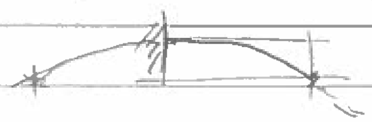
d) Actual $P_{cr} = \frac{\pi^2 EI}{(L_{eff})^2}$ where $L_{eff} = 2L$



$$= \left(\frac{\pi^2}{4}\right)\frac{EI}{L^2} = 2.47\frac{EI}{L^2}$$

so our approximation is not bad (3 as opposed to 2.47 factor).

Our approximation is incorrect because we used a parabolic mode shape $w \sim x^2$, whereas the true buckling mode shape (ie the eigenvector when $P=P_{cr}$) is cosinusoidal.



$$w \sim \left[1 - \cos\left(\frac{\pi x}{2L}\right)\right]$$

(10%)

$$\sim 1 - \left(1 - \frac{x^2}{2} + h.o.t\right) \sim x^2 + h.o.t.$$

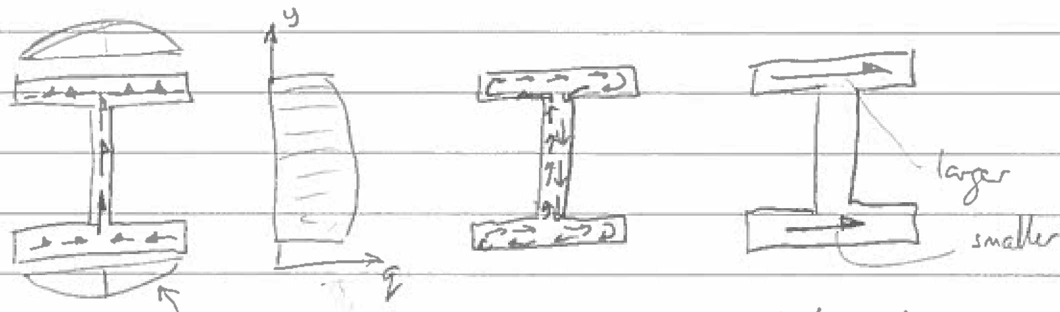
e) Can improve estimate by including more terms $(N>2) \Rightarrow K =$ matrix (total tangent stiffness matrix) and then look for conditions when K has a zero eigenvalue. (Or could just use a better mode shape eg. the exact one)!

(10%)

f) Expand $EI(x)$ as a polynomial $EI(x) = EI_0 \sum_{m=0}^{M} \beta_m x^m$ and include in $\frac{1}{2}\int_0^L EI(x)(w'')^2 dx$ calculations.

[10%]

Q4. a)



direct

from $q = \frac{S A_c \bar{y}}{I}$

St Venant.

Restrained Warping Torsion. (top flange in compression, so buckles (bends) more, so larger associated shear stress).

b) 406 x 140 x 46

$$M_{cr} = \underbrace{\frac{\pi}{L}\sqrt{E I_{min} G J}}_{M_{basic}}\left(1 + \frac{\pi^2}{L^2}\frac{E\Gamma}{GJ}\right)^{1/2}$$

406 x 140 x 46

$I_{min} = 538 \times 10^{-8}\ m^4$

$J = 19 \times 10^{-8}\ m^4$

$\Gamma = I_{minor}\frac{D_f^2}{4}$, $D_f = 403.2 - 11.2 = 392mm = 0.392m$

$= \frac{538 \times (0.392)^2}{4} = 20.7 \times 10^{-8}\ m^6$ also $G = \frac{E}{2.6}$

$M_{basic} = \frac{\pi}{L} E \sqrt{\frac{I_{min} J}{2.6}} = \frac{\pi}{8[m]}(210 \times 10^9)[N/m^2]\sqrt{\frac{(538)(19)}{2.6}} \times 10^{-8}$

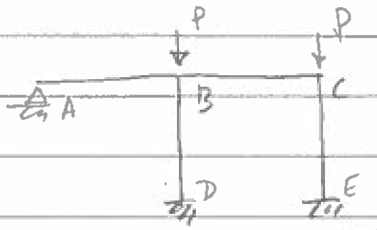
$= 51.7\ kNm$

Correction due to warping $= \left(1 + \left(\frac{\pi}{8}\right)^2 (2.6)\frac{20.7}{19}\right)^{1/2} = (1.437)^{1/2} = 1.20$

$\therefore M_{LTB} = 51.7 \times 1.2 = 62\ kNm$
EQ+OM

Q4 cont'd.

c) i)



By inspection.

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \overset{\text{from AB}}{3} + \overset{BD}{s} + \overset{\text{from BC}}{4} & \overset{\text{from BC}}{2} \\ \underset{BC}{2} & \underset{BC}{4} + \underset{CE}{s} \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

$$\therefore K_{TOT} = \frac{EI}{L} \begin{bmatrix} s+7 & 2 \\ 2 & s+4 \end{bmatrix}$$

ii) Characteristic eqn $(s+7)(s+4) - 4 = 0$

$$s^2 + 11s + 28 - 4 = 0$$

$$s^2 + 11s + 24 = 0$$

$$s = \frac{-11 \pm \sqrt{121 - 4(24)}}{2} = \frac{-11 \pm 5}{2} = \begin{cases} -3 \text{ or} \\ -8 \end{cases}$$

$\Rightarrow \frac{P}{P_E} \sim 2.73$ for $s = -3$ from graph.

$$\therefore P_{cr} \triangleq (2.73)\frac{\pi^2 EI}{L^2} \simeq 26.9 \frac{EI}{L^2}$$

iii) $s = -3 \Rightarrow \begin{bmatrix} M_B \\ M_C \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} -3+7 & 2 \\ 2 & -3+4 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$

so $\theta_B = -\theta_C/2$.

Also $\theta_A = -\theta_B/2 = \theta_C/4$.