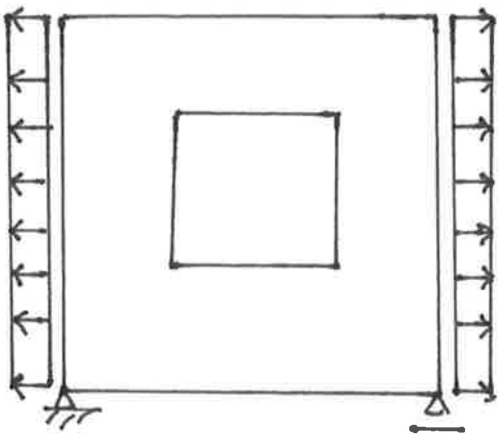
Qu 1

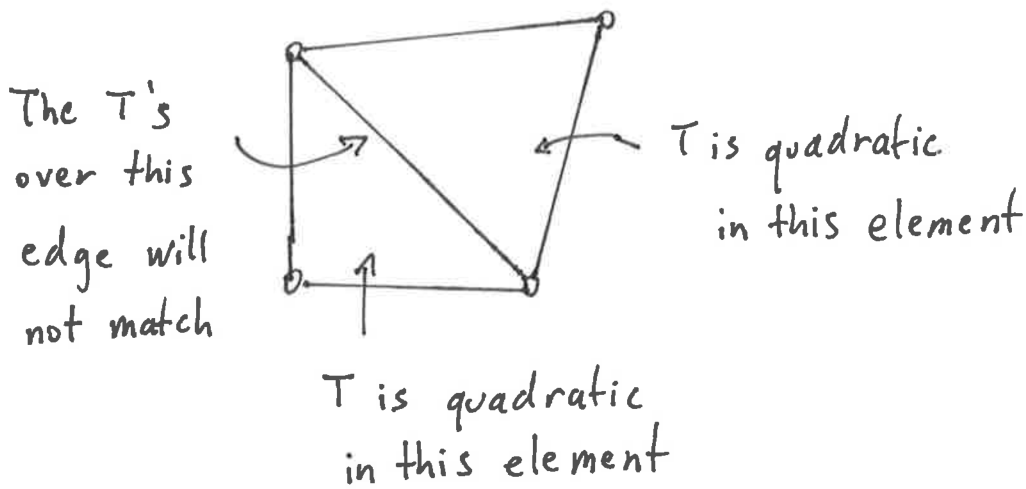
a) Supports are missing; system matrix is rank deficient and cannot be inverted. Need to add supports that do not alter the stress state.



b)

$$T^e(x,y) = \alpha_0^e + \alpha_1^e x^2 + \alpha_2^e y^2$$

- The linears are missing; will not converge. (cannot represent linear temperature distribution)
- The T's over the element boundaries will be not compatible.



c) i) $x = \cancel{N^1 x^1}^{=0} + N^2 x^2 + N^3 x^3 + \cancel{N^4 x^4}^{=0}$

$y = \cancel{N^1 y^1}^{=0} + \cancel{N^2 y^2}^{=0} + N^3 y^3 + N^4 y^4$

$\Longrightarrow$

$$4x = 2(1+\xi)(1-\eta) + a(1+\xi)(1+\eta)$$

$$4y = b(1+\xi)(1+\eta) + 2(1-\xi)(1+\eta)$$

ii)

$$4\frac{\partial x}{\partial \xi} = 2(1-\eta) + a(1+\eta)$$

$$4\frac{\partial x}{\partial \eta} = -2(1+\xi) + a(1+\xi)$$

$$4\frac{\partial y}{\partial \eta} = b(1+\xi) + 2(1-\xi)$$

$$4\frac{\partial y}{\partial \xi} = b(1+\eta) - 2(1+\eta)$$

at $\xi = 1, \eta = 1$

$4\frac{\partial x}{\partial \xi} = 2a \qquad 4\frac{\partial x}{\partial \eta} = -4+2a$

$4\frac{\partial y}{\partial \eta} = 2b \qquad 4\frac{\partial y}{\partial \xi} = 2b-4$

$$\det J = \frac{1}{16}\left(2a \cdot 2b - (2a-4)(2b-4)\right) > 0$$

$$\frac{ab}{4} - \left(\frac{ab}{4} - \frac{a}{2} - \frac{b}{2} + 1\right) > 0$$

$$\Longrightarrow \quad \frac{a}{2} + \frac{b}{2} - 1 > 0$$

$$\underline{a + b > +2}$$

d) Quadratic bubble

$$N_5 = (-\xi^2 + 1)(-\eta^2 + 1)$$

$$N_1 = \underbrace{\frac{1}{4}(1-\xi)(1-\eta)}_{\text{usual four node shape function}} - \frac{1}{4}\underbrace{(-\xi^2+1)(-\eta^2+1)}_{\text{quad. bubble}}$$

Q1. Examiner's Comment:

Overall this was a popular and well answered question. In part (a), most candidates did not see the missing boundary conditions and instead focused on the quality of the elements. For the suggested incomplete quadratic polynomial ansatz mostly the discontinuities across the element edges were mentioned. In part (c), although candidates usually computed the Jacobian matrix, quite a few did not recall that its determinant has to be larger than zero. In part (d), surprisingly many did not realise that it is the two-dimensional version of a similar one-dimensional question in the examples paper.

Qu 2

a) $T=0$ on Γ $\Rightarrow$ $w=0$ on Γ

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+c\,\frac{\partial T}{\partial x}+s=0$$

$$\cancel{\int \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x} w\right)}^{=0} = \int \frac{\partial^2 T}{\partial x^2} w + \int \frac{\partial T}{\partial x}\frac{\partial w}{\partial x}$$

$$\Rightarrow \quad -k\int\left(\frac{\partial T}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial T}{\partial y}\frac{\partial w}{\partial y}\right)+c\int\frac{\partial T}{\partial x}w+\int s w=0$$

b) $$c\int w\frac{\partial T}{\partial x}\,d\Omega$$

i) $N_1 = 1-\frac{x}{4}-\frac{y}{2}$ $\qquad N_2=\frac{x}{4}$ $\qquad N_3=\frac{y}{2}$

$$w=\left(1-\frac{x}{4}-\frac{y}{2}\right)w_1+\frac{x}{4}w_2+\frac{y}{2}w_3$$

$$T=\left(1-\frac{x}{4}-\frac{y}{2}\right)T_1+\frac{x}{4}T_2+\frac{y}{2}T_3$$

$$\frac{\partial T}{\partial x}=-\frac{T_1}{4}+\frac{T_2}{4}$$

(1,2) component of matrix

$$\int_\Omega \frac{1}{4}\left(1-\frac{x}{4}-\frac{y}{2}\right)=\int_0^4\int_0^{-\frac{x}{2}+2}\left(\frac{1}{4}-\frac{x}{16}-\frac{y}{8}\right)dy\,dx$$

$$=\frac{1}{3}$$

(2, 1) component of matrix

$$\int_\Omega \frac{x}{4}\left(-\frac{1}{4}\right) = \int_0^4 \int_0^{-\frac{x}{2}+2} -\frac{x}{16}\, dy\, dx$$

$$= -\frac{1}{3}$$

ii) Jacobian is constant for this element

$$c\int w \frac{\partial T}{\partial x}\, d\Omega = c\int w\Big(\frac{\partial T}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{\text{constant}} + \frac{\partial T}{\partial \eta}\cdot\underbrace{\frac{\partial \eta}{\partial x}}_{\text{constant}}\Big)\underbrace{J}_{\text{constant}}\, d\xi\, d\eta$$

Hence, we need to integrate terms of the form

$$c\int w \frac{\partial T}{\partial \xi} \quad \text{and} \quad c\int w \frac{\partial T}{\partial \eta}$$

For a square element w and T are of the form

$$w = \alpha_0 + \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \xi\eta$$

$$T = \beta_0 + \beta_1 \xi + \beta_2 \eta + \beta_3 \xi\eta$$

$$\frac{\partial T}{\partial \xi} = \beta_1 + \beta_3 \eta$$

$$\int w \frac{\partial T}{\partial \xi} = \int (\alpha + \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \xi\eta)(\beta_1 + \beta_3 \eta)\, d\xi\, d\eta$$

$\Longrightarrow$ highest term in η is quadratic

$\Longrightarrow$ 2 quadrature points in η

$\Longrightarrow$ highest term in ξ is linear

$\Longrightarrow$ 1 quadrature point in ξ

Q2. Examiner's Comment:

This was a straightforward and the most popular question. There were usually no problems in determining the weak form. Most candidates were confident in computing the requested matrix components in part (b). As to be expected there were quite a few arithmetic errors due to the length of the computations. Many candidates made a good attempt in answering the intentionally hard last part of this question. It was not very often mentioned that the Jacobian is constant and usually same number of quadrature points in both directions were suggested.

3) $\int_0^L \alpha v'' u'' \, dx + \int_0^L \beta v u \, dx + \gamma v (u-g)\Big|_L = 0$

a) Integrate by parts twice

$$-\int_0^L \alpha v' u''' + \alpha v' u''\Big|_0^L + \int_0^L \beta v u \, dx + \gamma v (u-g)\Big|_L = 0$$

$$\int_0^L \alpha v u'''' \, dx - \alpha v u'''\Big|_0^L + \alpha v' u''\Big|_0^L$$

$$+ \int_0^L \beta v u \, dx + \gamma v (u-g)\Big|_L = 0$$

$$\Rightarrow \quad \alpha \frac{d^4 u}{dx^4} + \beta u = 0 \qquad \text{on domain } (0, L)$$

Boundary conditions:

$x = 0$: $\quad v = 0 \quad$ or $\quad \alpha u''' = 0$

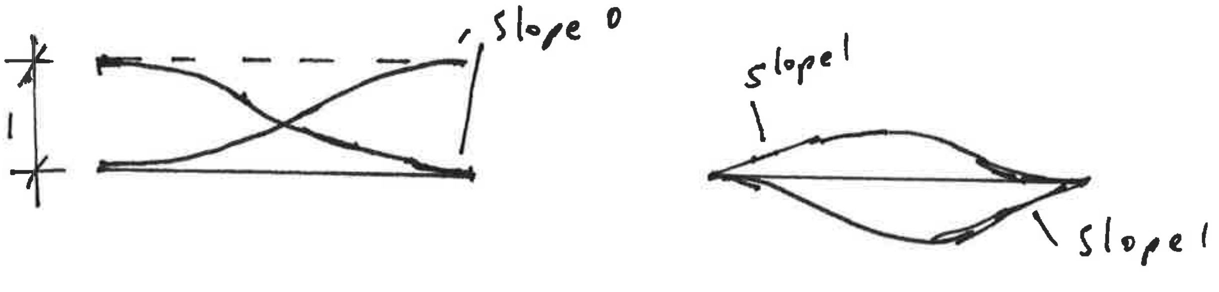
$\qquad\quad v' = 0 \quad$ or $\quad \alpha u'' = 0$

$x = L$: $\quad v' = 0 \quad$ or $\quad \alpha u'' = 0$

$\qquad\quad v = 0 \quad$ or $\quad -\alpha v + \gamma u = \gamma g$

b) Weak form contains 2nd order derivatives, therefore need shape functions with continuous first derivatives.

c)



d) Two nodes and four conditions

$$N = C_0 + C_1 x + C_2 x^2 + c_3 x^3$$

$$N' = c_1 + 2c_2 x + 3c_3 x^2$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

e) Shape functions are 3rd order polynomials

$\Rightarrow \int \beta v u \, dx$ term has degree 6

$\Rightarrow$ 4 points (can integrate $2(4)-1=7$ exactly)

Q3. Examiner's Comment:

Although this question was in general well answered, surprisingly many had in part (a) problems with systematically deriving the strong form. Most resorted to guessing the strong form, which is, of course, not sufficient for identifying the correct boundary conditions. Parts (b) and (c) on Hermitian shape functions were in general well answered. In the last part, most gave the number of quadrature points for integrating the energy instead of the elastic support term.

4| a) $M\ddot{a} + \alpha K\dot{a} + Ka = f$

Verlet scheme:

$$a_{n+1} = a_n + \Delta t\,\dot{a}_n + \frac{\Delta t^2}{2}\ddot{a}_n \qquad (1)$$

Shift index by one for velocity

$$\dot{a}_n = \dot{a}_{n-1} + \frac{\Delta t}{2}\left(\ddot{a}_n + \ddot{a}_{n-1}\right) \qquad (2)$$

Express system at t_n

$$M\ddot{a}_n + \alpha K\dot{a}_n + Ka_n + f_n$$

insert expressions for $\ddot{a}_n$ and $\dot{a}_n$

From (1)

$$\ddot{a}_n = \frac{2}{\Delta t^2}\left(a_{n+1} - a_n - \Delta t\,\dot{a}_n\right)$$

$$\Longrightarrow \quad M\frac{2}{\Delta t^2}\left(a_{n+1} - a_n - \Delta t\,\dot{a}_n\right)$$
$$+ \alpha K\left(\dot{a}_{n-1} + \frac{\Delta t}{2}\left(\ddot{a}_n + \ddot{a}_{n-1}\right)\right)$$
$$+ Ka_n = f_n$$

Rearrange:

$$\frac{2}{\Delta t^2}Ma_{n+1} = f_n + \frac{2}{\Delta t^2}M\left(a_n + \Delta t\,\dot{a}_n\right)$$
$$- \alpha K\left(\dot{a}_{n-1} + \frac{\Delta t}{2}\left(\ddot{a}_n + \ddot{a}_{n-1}\right)\right)$$
$$- Ka_n$$

b) If $\alpha > 0$ model has diffusive component. ②

Verlet scheme is explicit, so critical time step may be problematic. With $\alpha > 0$ implicit scheme probably better.

c) Since scheme is explicit lumped mass matrix is appropriate to avoid solving system of coupled equations.

Could lumped matrix by row-sum of the consistent mass matrix or nodal integration

d) 2D:

Doubling number of elements in each direction gives $2^2 = 4$ increase in number of dof

Solver is $O(n^2) \rightarrow O((4n)^2) \rightarrow$ factor 16

3D:

Doubling number of elements in each direction gives $2^3 = 8$ increase in number of dof

$O(n^2) \rightarrow O((8n)^2) \rightarrow$ factor 64 increase

Q4. Examiner's Comment:

This was the least popular question and was relatively hard despite the high average. A common source of error was that most candidates ended up with an implicit integration scheme in part (a). The part (c) on the lumped mass matrix was usually very well answered. Surprisingly, in part (d) many were not able to determine the correct scaling of the solution time, notwithstanding the similarities to questions in the last examples paper.