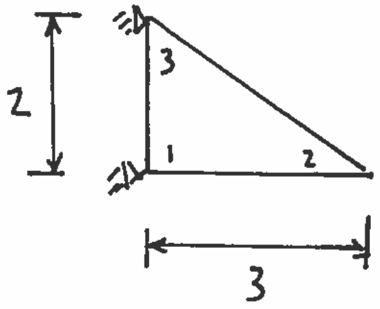
1a)



$$N_2 = \frac{x}{3}$$

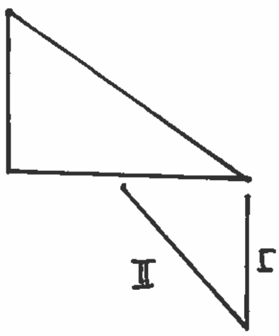
$$B = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$D = \frac{200}{1-0{,}3^2} \begin{bmatrix} 1 & 0{,}3 & 0 \\ 0{,}3 & 1 & 0 \\ 0 & 0 & 0{,}35 \end{bmatrix} = 219{,}8 \begin{bmatrix} 1 & 0{,}3 & 0 \\ 0{,}3 & 1 & 0 \\ 0 & 0 & 0{,}35 \end{bmatrix}$$

$$K = 219{,}8 \cdot 3 \underbrace{\underbrace{\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0{,}3 & 0 \\ 0{,}3 & 1 & 0 \\ 0 & 0 & 0{,}35 \end{bmatrix}}_{\begin{bmatrix} \frac{1}{3} & \frac{0{,}3}{3} & 0 \\ 0 & 0 & \frac{0{,}35}{3} \end{bmatrix}} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}}_{\begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{0{,}35}{9} \end{bmatrix}}$$

$$K = \begin{bmatrix} 73.27 & 0 \\ 0 & 25.64 \end{bmatrix}$$

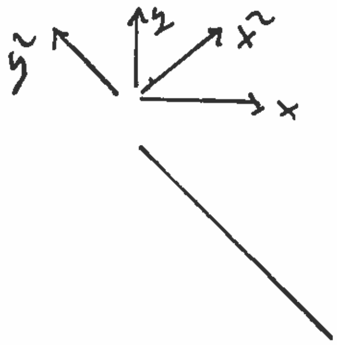
b)



$$K^{I} = \frac{EA}{1.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 66.7 & -66.7 \\ -66.7 & 66.7 \end{bmatrix}$$

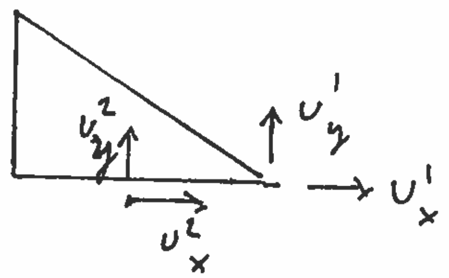
$$K^{II} = \frac{EA}{2.12}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 47.17 & -47.17 \\ -47.17 & 47.17 \end{bmatrix}$$

Consider bar II



$$\begin{bmatrix} u_{\tilde{x}} \\ u_{\tilde{y}} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\underbrace{\underbrace{\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 47.17 \end{bmatrix}}_{\begin{bmatrix} 0 & -33.35 \\ 0 & 33.35 \end{bmatrix}} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{\begin{bmatrix} 23.59 & -23.59 \\ -23.59 & 23.59 \end{bmatrix}}$$



Global stiffness

$$\begin{pmatrix} 73.27 & 0 & 0 & 0 \\ 0 & 25.64+66.7 & 0 & 0 \\ 0 & 0 & 23.59 & -23.59 \\ 0 & 0 & -23.59 & 23.59 \end{pmatrix} \begin{pmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f_y \\ 0 \\ 0 \end{pmatrix}$$

Note that $u_x^2 = \frac{1}{2} u_x^1$

$u_y^2 = \frac{1}{2} u_y^1$

leading to $\Rightarrow$ $\begin{pmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_x^1 \\ u_y^1 \end{pmatrix}$

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 73.27 & 0 & 0 & 0 \\ 0 & 92.34 & 0 & 0 \\ 0 & 0 & 23.59 & -23.59 \\ 0 & 0 & -23.59 & 23.59 \end{pmatrix}}_{\begin{pmatrix} 73.27 & 0 & 11.80 & -11.80 \\ 0 & 92.34 & -11.80 & 11.80 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}}_{\begin{pmatrix} 79.17 & -5.80 \\ -5.80 & 98.24 \end{pmatrix}}$$

$$\begin{pmatrix} 79.17 & -5.80 \\ -5.80 & 98.24 \end{pmatrix} \begin{pmatrix} u_x^1 \\ u_y^1 \end{pmatrix} = \begin{pmatrix} 0 \\ f_y \end{pmatrix}$$

$\Rightarrow u_x^1 = 7.49 \cdot 10^{-4} \quad u_y^1 = 1.02 \cdot 10^{-2}$

2) a) Too few integration points lead to zero energy modes. Need to increase number of integration points.

b) $$\int (u^h - f)\, w^h \, d\Omega = 0$$

$$\int \Big(\sum_I N_I u_I - f \Big) \sum_J N_J w_J \, d\Omega = 0$$

$$\sum_J w_J \underbrace{\Big(\sum_I \int N_I N_J u_I \quad - f N_J \Big)}_{=0 \text{ for arbitrary } w_I} = 0$$

$\Longrightarrow$

$$\sum_J \sum_I \int N_I N_J u_I = \sum_J \int f N_J$$

$$M_{IJ} = \int N_I N_J \qquad F_J = \int f N_J$$

$$\Longrightarrow \quad \underline{M}\, \underline{u} = \underline{F}$$

'mass matrix' 'force vector'

This can also be obtained from

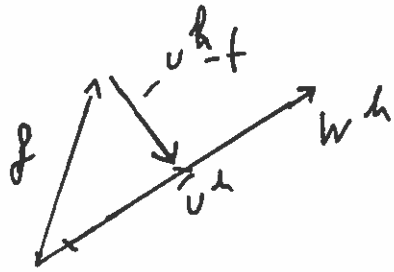
$$\pi = \frac{1}{2} \int \Big(\sum_I N_I u_I - f \Big)^2 \qquad \text{'least squares fit'}$$

<u>minimize π</u>

$$\frac{\partial \pi}{\partial u_J} = \int \Big(\sum_I N_I u_I - f \Big) N_J = 0$$

$$\int w^h \underbrace{(u^h - f)} = 0$$

error orthogonal
to all possible
Finite element
test functions



$\Rightarrow$ hence u^h is a projection

c)

$$\int u_{,i} w_{,i} + \int u\, c_i\, w_{,i} + \int (u - f) w = 0$$

$$-\int u_{,ii}\, w + BT - \int (u\, c_i)_{,i}\, w + BT \int (u - f) w = 0$$

$$-u_{,ii} - (u c_i)_{,i} + u - f = 0$$

Q1.

1. (a) (i) $-d/dx(E\,du/dx) = f$

Multiply by test function v & integrate.

$$-\int_0^L v \frac{d}{dx}\left(E\frac{du}{dx}\right) dx = \int_0^L v f \, dx$$

Integrate by parts:

$$\int_0^L \frac{dv}{dx} E \frac{du}{dx}\, dx - v \cancel{\frac{d}{dx}}\left(E\frac{du}{dx}\right)\Big|_0^L = \int_0^L v f\, dx$$

- Dirichlet b.c. @ $x=0$ ∴ $v(0) = 0$
- Insert Neumann b.c. @ $x = L$

$$\Rightarrow \int_0^L \frac{dv}{dx} E \frac{du}{dx}\, dx = \int_0^L v f \, dx + v h\Big|_{x=L}$$

for all admissible v

(ii) $\underline{\underline{k}}_e = \cancel{EA}\ \frac{E}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\underline{f}_e = f\begin{bmatrix} \int_0^l v\, dx \\ \int_0^l v\, dx \end{bmatrix} = \frac{f}{2}\begin{bmatrix} l \\ l \end{bmatrix}$$

(iii) $\cancel{\underline{f}_e} = \int \underbrace{v f}\, dx$ — polynomial degree of integral $= 1$

$\Rightarrow$ 1 quadrature point for exact integration.

(b) (i)

$$+\int_0^L \frac{dv}{dx} E \frac{du}{dx} \quad g - v E \frac{du}{dx}\Big|_{x=L} = \int_0^L v f dx$$

Re-arrang Robin bc:

$$au + E\frac{du}{dx} = g \implies E\frac{du}{dx} = g - au$$

$$\int_0^L \frac{dv}{dx} E \frac{du}{dx} + vau\Big|_{x=L} = \int_0^L vf dx + vg\Big|_{x=L}$$

(ii) $$\underline{\underline{k}}_e = \frac{E}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + a\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{f}}_e = \frac{f\ell}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

(iii) force/stress $= g - au$

$= a(g/a - u)$

$\therefore$ a ~~acts~~ is effectively a spring stiffness

Q2

(a) $$\frac{y_{n+1} - y_n}{\Delta t} = -\lambda((1-\theta)y_n + \theta y_{n+1})$$

Re-arrang into $y_{n+1} = A y_n$, when A is the amplification factor:

$$\cancel{\text{H}}\ \cancel{y_{n+1}}\ (1 + \lambda\Delta t\theta)y_{n+1} = (1 \mp \lambda\Delta t(1-\theta))y_n$$

$$y_{n+1} = \underbrace{\frac{(1 + \lambda\Delta t\theta - \lambda\Delta t)}{(1 + \lambda\Delta t\theta)}}_{A} y_n$$

For stability, $|A| \leq 1$

$$A = \underbrace{\frac{1 + \lambda\Delta t\theta}{1 + \lambda\Delta t\theta}}_{=1} - \underbrace{\frac{\lambda\Delta t}{1 + \lambda\Delta t\theta}}_{\text{must be } < 2 \text{ for all } \Delta t \text{ for } |A| \leq 1}$$

$$\Rightarrow \theta \geq \frac{1}{2} \Rightarrow < 2 \text{ for all } \Delta t$$

(b) (i) $$\underline{\underline{M}}\,\dot{\underline{u}} + \underline{\underline{K}}\,\underline{u} = \underline{f}$$

$$\underline{\underline{M}}\left(\frac{\underline{u}_{n+1} - \underline{u}_n}{\Delta t}\right) + \underline{\underline{K}}(\cancel{\theta}(1-\theta)\underline{u}_n + \theta\underline{u}_{n+1}) = \underline{f}_{n+\theta}$$

$(1-\theta)$

$$\underline{\underline{M}}(\underline{u}_{n+1}) - \underline{\underline{M}}\,\underline{y}_n + \Delta t\,\underline{\underline{K}}\,\underline{y}_n + \Delta t\theta\,\underline{\underline{K}}\,\underline{u}_{n+1} = \Delta t\,\underline{f}_{n+\theta}$$

$$(\underline{\underline{M}} + \theta\Delta t\,\underline{\underline{K}})\underline{u}_{n+1} = \Delta t\,\underline{f}_{n+\theta} + \underline{\underline{M}}\,\underline{u}_n - \Delta t(1-\theta)\underline{\underline{K}}\,\underline{u}_n$$

$$\hat{\underline{\underline{M}}}\,\underline{u}_{n+1} = \hat{\underline{f}}$$

(ii) Modal analysis reduces two dof problem to two eqns with same form as model problem, wh λ is an eigenvalue of:

$$\underline{\underline{K}}\,\underline{x} = \lambda\underline{\underline{M}}\,\underline{x}$$

Q.2

To find λ:

$$(\underline{\underline{K}} - \lambda \underline{\underline{M}})\underline{x} = \underline{0}$$

$$\Rightarrow \det(\underbrace{\underline{\underline{K}} - \lambda \underline{\underline{M}}}_{\underline{\underline{K}}'}) = 0$$

$$|\underline{\underline{K}}'| = \begin{vmatrix} 2-6\lambda & -1-\lambda \\ -1-\lambda & 1-5\lambda \end{vmatrix}$$

$$= (2-6\lambda)(1-5\lambda) - (-1-\lambda)^2$$

$$= 2 - 16\lambda + 30\lambda^2 - (1 + 2\lambda + \lambda^2)$$

$$= 1 - 18\lambda + 29\lambda^2$$

Find roots:

$$\frac{18 \pm \sqrt{18^2 - 4(29)}}{2(29)} = \frac{18 \pm \sqrt{208}}{58}$$

$\Delta t_{crit} = \frac{2}{\lambda_i}$ $\therefore$ λ_{max} determ. step

$$\Delta t_{crit} = 2/\lambda_{max} \cong 2(58)/(18 \pm \sqrt{208})$$

$$\cong \underline{3.58}$$

(iii) When $\theta < 1/2$, critical time step is proportional to $1/\lambda_{max}$. $\lambda_{max} \propto 1/h^2$ (cell size)

$\therefore$ $\Delta t_{crit} \propto h^2$

This is often too restrictive for practical use, therefore an implicit scheme is often preferable.