1.1 (i) $-EAu'' = -\alpha u$

Multiply by test function v and integrate

$$-\int_0^L EA\, v u''\, dx + \int_0^L \alpha v u\, du$$

Integration by parts:

$$\int_0^L EA\, v' u'\, dx - v EA u' \Big|_0^L + \int \alpha v u\, dx$$

Since $u = 0$ at $x = 0$, $v(0) = 0$, Using this and insert the Neumann (force) boundary condition.

$$\int_0^L (EA\, u' v' + \alpha u v)\, dx = vF \big|_{x=L}$$

(ii) To integrate exactly, term $\alpha v u$ has highest degree. Use Gauss quadrature, which with n points can integrate a polynomial of degree $2n-1$ exactly. Therefore.

- P1 element: use 2 points (integrand is polynomial degree 2
- P2 element: use 3 points (integrand is polynomial degree 4

(iii) In a finite element implementation, ~~zero~~ set rows and columns for degree of freedom associated with Dirichlet boundary condition and set 1 on correspondin diagonal terms. Or remove the associated rows and columns from the system.

(1.2) (i)

Left-hand side

$$\int_0^{L/3} \left(EA\, v'u' \cancel{dx}\ \alpha uv\right) dx = v^-(u^+-u^-)\,k\Big|_{L/3}$$

Right-hand side

$$\int_{L/3}^{L} (EA\, v'u' + \alpha uv)\, dx = -\, v^+(u^+-u^-)\,k\Big|_{L/3} + vF\Big|_L$$

Combining :

$$\int_0^L \left(EA\, u'v' + \alpha uv\right) + k(v^+-v^-)(u^+-u^-)\Big|_{x=L/3}$$

$$= vF\,\Big|_L .$$

(ii) If $\alpha = k = 0$, system would be singular.

Q2

(a) (i) A priori error estimate : $\|e\| \leq C h^{p+1} |u|_r$

$|u|_r$: exact soln

p: polynomial degree

h_1: cell size original ~~matort~~ mesh

h_2: cell size refined mesh

To halve error: $\frac{h_1^{p+1}}{2} = h_2^{p+1}$

$$\therefore \quad h_2 = \frac{h_1}{2^{\left(\frac{1}{p+1}\right)}}$$

linear elements ($p=1$): $h_2 = \frac{1}{\sqrt{2}} h_1$

quadrat element ($p=2$) $h_2 = \frac{1}{\sqrt[3]{2}} h_1$

Same in 2D & 3D.

(ii) 3D: $n \propto \frac{1}{h^3}$ $\therefore$ cost $O\left(\frac{1}{h^6}\right)$

$n_2 = \frac{1}{h_2^3} = \frac{2}{h_1^3}$ $\therefore$ cost $O\left(\frac{4}{h_1^6}\right)$

$\therefore$ Cost increase by factor of $\underline{4}$

(iii) A priori estimate rely on sufficiently smooth exact solution. If not smooth enough, increasing element degree will not reduce error

Q 2 (cont)

(b) (i) Assembly cost is proportional to number of elements. Number of sets dofs (n) is proport to numb of element.

$\therefore$ cost will double

(ii) In 3D, increasin cell count by facto of 8 correspd to halving cell size,

$$h_2 = \frac{h_1}{2}$$

$\therefore$ Critic time step $\propto h^2$, $\therefore$ time step must be redu by 4.

Cost per time step increa by fact of 8, and 4 time th numb of steps required.

$\therefore$ Increa in cost is $4 \cdot 8 = \underline{32}$

(iii) The time step restrict of $\Delta t_{cr} \propto t^2$ is very strong. Implicit method, which can be constructd to have no time step restriction, would be better.

3.1

(a) $x = 2 + \xi(5-2) + \eta(3-2) = 2 + 3\xi + \eta$

$y = 1 + \xi(3-1) + \eta(4-1) = 1 + 2\xi + 3\eta$

for $\xi = \frac{1}{3}$, $\eta = \frac{1}{3}$

P : $\boxed{\begin{array}{l} x = 2 + 1 + \frac{1}{3} = \frac{10}{3} \\ y = 1 + \frac{2}{3} + 1 = \frac{8}{3} \end{array}}$

(b) $\underline{\underline{J}} = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$, $\det(\underline{J}) = 7$, the area of the triangle is $A = \frac{7}{2}$

(c) $\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi}$

$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial y}{\partial \eta}$

$$\begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix}$$

3.2

(d) $N_5 = 4\eta\xi$

$$x = 2 + 3\xi + \eta \qquad (1)$$

$$y = 1 + 2\xi + 3\eta \qquad (2)$$

$3\times(1) - (2)$ $\quad 3x - y = 5 + 7\xi$

$$\xi = \frac{1}{7}(3x - y - 5)$$

$2\times(1) - 3\times(2)$ $\quad 2x - 3y = 1 - 7\eta$

$$\eta = \frac{1}{7}(3y + 1 - 2x)$$

Hence $\quad N_5 = \frac{4}{49}(3x - y - 5)(3y + 1 - 2x)$

(e) $\quad N_4 = 4\xi(1 - \xi - \eta)$

$$\frac{\partial N_4}{\partial \xi} = 4 - 8\xi - 4\eta \quad , \quad \frac{\partial N_4}{\partial \eta} = -4\xi$$

$$\frac{\partial N_4}{\partial x} = \frac{3}{7}\frac{\partial N_4}{\partial \xi} - \frac{2}{7}\frac{\partial N_4}{\partial \eta} = \frac{3}{7}(4 - 8\xi - 4\eta) - \frac{2}{7}\times 4\xi$$

$$= \cancel{47}\ \frac{12}{7} - \frac{32}{7}\xi - \frac{12}{7}\eta$$

$$\int_K \frac{\partial N_4}{\partial x}\cdot\frac{\partial N_4}{\partial x}\,dx\,dy = \int_{\hat{K}} \frac{\partial N_4}{\partial x}\cdot\frac{\partial N_4}{\partial x}\det(J)\,d\xi\,d\eta.$$

$$= \left(\frac{\partial N_4}{\partial x}\right)^2\left(\frac{1}{3},\frac{1}{3}\right)\times 7 \times \frac{1}{2} = 0.508$$

$$\frac{\partial N_4}{\partial x}\left(\frac{1}{3},\frac{1}{3}\right) = -0.381$$

(4.1)

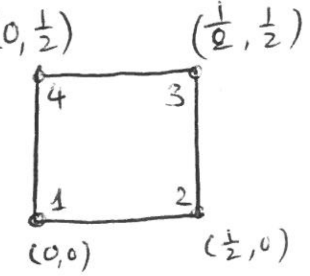
(a) Let the square is noted as Ω and v a ~~test~~ function where $v=0$ at $\partial\Omega$.

$$\int_\Omega v \, \nabla^2 u \, dx dy = -\int_\Omega v \, dx dy$$

Integrate by part

$$\int_\Omega \nabla u \, \nabla v \, dx \, dy = \cancel{\int_{\partial\Omega} \frac{\partial u}{\partial n} \overset{0}{\overset{\shortparallel}{v}} \, dl} + \int_\Omega v \, dx dy.$$

(b) $N_1^e = 4(\tfrac{1}{2}-x)(\tfrac{1}{2}-y)$



$N_2^e = 4x(\tfrac{1}{2}-y)$

$N_3^e = 4xy$

$N_4^e = 4(\tfrac{1}{2}-x)y$

(i)

$$\underline{\underline{B}}^e = \begin{bmatrix} \frac{\partial \underline{N}^e}{\partial x} \\ \frac{\partial \underline{N}^e}{\partial y} \end{bmatrix} = \begin{bmatrix} -2+4y & 2-4y & 4y & -4y \\ -2+4x & -4x & 4x & 2-4y \end{bmatrix}$$

(ii)

$$K_{11} = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} + \frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y} \, dx dy = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} (-2+4y)^2 + (-2+4x)^2 dx dy =$$

$$K_{12} = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} (-2+4y)(2-4y) + (2-4x)(4x) \, dx dy = -\frac{1}{3}$$

$$K_{13} = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} (-2+4y)(4y) + (-2+4x)(4x) \, dx dy = -\frac{2}{3}$$

$$K_{14} = K_{12} = -\frac{1}{3} \quad \text{(by symmetry)}.$$

We can complete the matrix $\boxed{\text{conductant}}$ ~~mat~~ by symmetry.

$$\underline{\underline{K}}^e = \frac{1}{3}\begin{bmatrix} 4 & -1 & -2 & -1 \\ & 4 & -1 & -2 \\ & & 4 & -1 \\ & & & 4 \end{bmatrix}$$

(4.2)

(b) (iii) $f_3 = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 4xy \, dx dy = 4 \int_0^{1/2} x \, dx \int_0^{1/2} y \, dy = 4 \left. \frac{x^2}{2} \right|_0^{\frac{1}{2}} \left. \frac{y^2}{2} \right|_0^{1/2} = \frac{1}{16}$

by symmetry $\quad f_1 = f_2 = f_4 = \frac{1}{16}.$

(c) The only unknown is at node 5. The equation for node 5 is

$$-\frac{2}{3}u_1 - \frac{2}{3}u_2 - \frac{2}{3}u_3 - \frac{2}{3}u_4$$

$$+ \frac{16}{3}u_5 - \frac{2}{3}u_6 - \frac{2}{3}u_7 - \frac{2}{3}u_8 - \frac{2}{3}u_9 = 4 \times \frac{1}{16}$$

$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$
$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$
$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{1}{3}$
$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$

Appling boundary condition $u_1 = \cdots = u_4 = 0$

$u_6 = \frac{1}{4}, \quad u_7 = u_8 = u_9 = 0,$

$\cancel{u_5 = \frac{3}{64}} \qquad \frac{16}{3}u_5 - \frac{2}{3} \times \frac{1}{4} = 4 \times \frac{1}{16}$

$$u_5 = \frac{9}{64}$$