3D7 2019

Q1

a)

Weak form:
$$\int_{x_1}^{x_2} \frac{dv}{dx} EA \frac{du}{dx}\, dx = \int_{x_1}^{x_2} v f\, dx + \underbrace{v EA \frac{du}{dx}\Big|_{x_1}^{x_2}}$$

$v(x_2) = 0$ (Dirichlet condition) $\qquad = v EA \frac{du}{dx} n\Big|_{x_2} + \underbrace{v EA \frac{du}{dx} n}_{h}\Big|_{x_1}$

At x_1: $EA \frac{du}{dx} \underset{=-1}{n} = h$

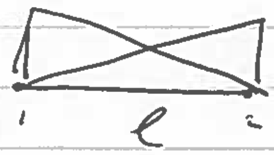
$$\Rightarrow \int_{x_1}^{x_2} \frac{dv}{dx} EA \frac{du}{dx}\, dx = \int_{x_1}^{x_2} v f\, dx + vh\Big|_{x_1}.$$

b i
$$-\frac{d}{dx}\left(EA\left(\frac{du}{dx} - \varepsilon_T\right)\right) = f$$

$$\Rightarrow \int_{x_1}^{x_2} \frac{dv}{dx} EA \frac{du}{dx}\, dx = \int_{x_1}^{x_2} v f + \int_{x_1}^{x_2} \frac{dv}{dx} EA\, \varepsilon_T\, dx$$
$\varepsilon_T = \alpha(T - T_0)$

$v(x_1) = v(x_2) = 0$ (Dirichlet bcs)

ii



$$\underline{b} = \int_{x_1^e}^{x_2^e} \begin{bmatrix} -\frac{1}{\ell} EA\alpha(T-T_0) \\ \frac{1}{\ell} EA\alpha(T-T_0) \end{bmatrix} dx$$

$$= \begin{bmatrix} -EA\alpha(T-T_0) \\ EA\alpha(T-T_0) \end{bmatrix}$$

iii Integrand is linear → use 1 quadrature point

c i $x = \sum_{i=1}^{3} N_i(\xi) x_i^e \qquad x(\xi=0) = x_3^e$

ii $\frac{du}{dx} = \sum_{i=1}^{3} \frac{dN(\xi)}{dx} u_i$

$$\frac{dN}{d\xi} = \frac{dN}{dx}\frac{dx}{d\xi} \implies \frac{dN}{dx} = \frac{dN}{d\xi}\left(\frac{dx}{d\xi}\right)^{-1}$$

		$dN/d\xi$
$N_1 =$	$\frac{1}{2}\xi^2 - \frac{1}{2}\xi$	$\xi - 1/2$
$N_2 =$	$\frac{1}{2}\xi^2 + 1/2\,\xi$	$\xi + 1/2$
$N_3 =$	$-\xi^2 + 1$	-2ξ

$$\frac{dx}{d\xi} = (\xi - 1/2)x_1^e + (\xi + \tfrac{1}{2})x_2^e - 2\xi x_3^e$$

$$\frac{du(\xi)}{dx} = \frac{(\xi - \frac{1}{2})u_1^e + (\xi + \frac{1}{2})u_2^e - 2\xi u_3^e}{(\xi - \frac{1}{2})x_1^e + (\xi + 1/2) - 2\xi x_3^e}$$

Q2

a) Include inertia and self-weight

$$\rho \ddot{v} = f + \rho g - \frac{d^2}{dx^2}\left(EI \frac{d^2 v}{dx^2}\right)$$

Adding reaction force from elastic foundation: $f = -kv$

$$\Rightarrow \rho \ddot{v} + kv + \frac{d^2}{dx^2}\left(EI\frac{d^2 v}{dx^2}\right) = \rho g$$

Multiply by weight function and integrate by parts

1. $$\int_L w \rho \ddot{v}\, dx + \int_L w k v\, dx - \int_L \frac{dw}{dx} \frac{d}{dx}\left(EI\frac{d^2 v}{dx^2}\right) dx + \cancel{w \frac{d}{dx}\left(EI\frac{d^2 v}{dx^2}\right)\Bigg|_{x_1}^{x_2}}$$

$$= \int_L w \rho g\, dx$$

(= 0 : no shear force at ends)

2. $$\int_L w \rho \ddot{v}\, dx + \int_L w k v\, dx + \int_L \frac{d^2 w}{dx^2} EI \frac{d^2 v}{dx^2}\, dx - \cancel{\frac{dw}{dx} EI \frac{d^2 v}{dx^2}\Bigg|_{x_1}^{x_2}} = \int_L w \rho g\, dx$$

zero bending moment applied

$$\Rightarrow \int_L w \rho \ddot{v}\, dx + \int_L w k v\, dx + \int_L \frac{d^2 w}{dx^2} EI \frac{d^2 v}{dx^2} = \int_L w \rho g\, dx \quad \forall w$$

b Weak form contains 2nd order derivatives, therefore need C^1 basis, e.g. Hermite polynomials

c Could model with point forces (moving) at wheels.
Mathematically: add $\int_L w \delta F\, dx$ to RHS vector
↑ Delta function

d) From the $u_{n+1} = u_n + \cdots$ equation,

$$\dot{u}_n = (u_{n+1} - u_n)\Delta t - \ddot{u}_n \Delta t / 2 \qquad (1)$$

Take the $\dot{u}_{n+1} = \dot{u}_n + \cdots$ equation and shift the index by 1:

$$\dot{u}_n = \dot{u}_{n-1} + \tfrac{1}{2}(\ddot{u}_{n-1} + \ddot{u}_n)\Delta t \qquad (2)$$

Insert (1) into (2) (with appropriate shift):

$$\frac{u_{n+1} - u_n}{\Delta t} - \frac{1}{2}\ddot{u}_n \Delta t = \frac{u_n - u_{n-1}}{\Delta t} - \frac{1}{2}\ddot{u}_{n-1}\Delta t + \frac{1}{2}(\ddot{u}_{n-1} + \ddot{u}_n)\Delta t$$

Re-arranging:

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta t^2} = \ddot{u}_n \qquad (3)$$

Express linear system at t_n (scheme in (3) is explicit)

$$\underline{\underline{M}}\,\ddot{\underline{u}}_n + \underline{\underline{K}}\,\underline{u}_n = \underline{f}_n$$

Insert (3):

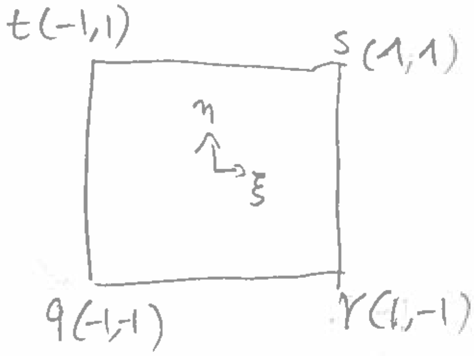
$$\frac{1}{\Delta t^2}\underline{\underline{M}}(\underline{u}_{n+1} - 2\underline{u}_n + \underline{u}_{n-1}) + \underline{\underline{K}}\,\underline{u}_n = \underline{f}_n$$

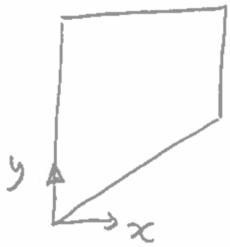
$$\Longrightarrow \underline{\underline{M}}\,\underline{u}_{n+1} = \Delta t^2(\underline{f}_n - \underline{\underline{K}}\,\underline{u}_n) + \underline{\underline{M}}(2\underline{u}_n - \underline{u}_{n-1})$$

Solution of system of equations can be avoided if $\underline{\underline{M}}$ is made diagonal, e.g. via mass lumping

Scheme is explicit and therefore has critical time step. Critical time step is very restrictive for 4th order equations $\longrightarrow$ explicit is not usually practical in such cases.

3) (1-1)





(a) shape functions

$n_q = (1-\xi)(1-\eta)/4 \qquad n_s = (1+\xi)(1+\eta)/4$

$n_r = (1+\xi)(1-\eta)/4 \qquad n_t = (1-\xi)(1+\eta)/4$

(b) iso-parametric mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} n_q & 0 & n_r & 0 & n_s & 0 & n_t & 0 \\ 0 & n_q & 0 & n_r & 0 & n_s & 0 & n_t \end{bmatrix}}_{\underline{\underline{N}}} \begin{bmatrix} x_Q \\ y_Q \\ x_R \\ y_R \\ x_S \\ y_S \\ x_T \\ y_T \end{bmatrix} \quad \text{where} \begin{bmatrix} x_Q \\ y_Q \\ \vdots \\ \vdots \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2n_r + 2n_s \\ n_r + 2n_s + 2n_t \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2(1+\xi)(1-\eta) + 2(1+\xi)(1+\eta) \\ (1+\xi)(1-\eta) + 2(1+\xi)(1+\eta) + 2(1-\xi)(1+\eta) \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 + 4\xi \\ 5 + \xi + 3\eta - \xi\eta \end{pmatrix}$$

(c) $p = (\frac{1}{2}, \frac{1}{2})$

$$\begin{bmatrix} x \\ y \end{bmatrix}_P = \frac{1}{4}\begin{pmatrix} 4 + 4 \times \frac{1}{2} \\ 5 + \frac{1}{2} + \frac{3}{2} - \frac{1}{2} \times \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{27}{16} \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.6875 \end{pmatrix}$$

(1-2)

(d) (i) for line rs, $\xi = 1$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 \\ 6+2\eta \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2} + \frac{\eta}{2} \end{pmatrix}$$ straight line, coincides with RS.

(ii) for straight line qps, $\eta = \xi$, $\xi = (-1, 1)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 + 4\xi \\ 5 + 4\xi - \xi^2 \end{pmatrix}$$ ← non-linear term;

Line QPS is not a straight line.

For a straight line passing Q and R,

$y = 2x$, but $x_p = \frac{3}{2}$, $y_p = \frac{27}{4}$, $y_p \neq 2x_p$

(e) The same matrix $\underset{=}{N}$ is used to transform displacements.

so $$\begin{bmatrix} u \\ v \end{bmatrix} = \underset{\sim}{N} \begin{bmatrix} 0.2 \\ 0.1 \\ 0 \\ 0.15 \\ 0 \\ 0 \\ -0.05 \\ 0 \end{bmatrix} = \begin{pmatrix} 0.2\, n_q - 0.05\, n_t \\ 0.1\, n_q + 0.15\, n_r \end{pmatrix}$$

for $\xi = \frac{1}{2}, \eta = \frac{1}{2}$

$n_q = \frac{1}{16}$

$$= \begin{pmatrix} 0.2/16 - 0.05 \cdot \frac{3}{16} \\ \frac{0.1}{16} + \frac{0.15 \times 3}{16} \end{pmatrix} = \begin{pmatrix} \frac{1}{320} \\ \frac{11}{320} \end{pmatrix}$$

$n_r = \frac{3}{16}$

$n_t = \frac{3}{16}$

4) (2-1)

Heat equation

(a) $$\begin{cases} \nabla \cdot (\underline{\underline{D}} \nabla T) + S = 0 \quad \text{with B.Cs.} \\ x = 0, \quad T = 1 \\ x = 2, \quad \cancel{\frac{\partial T}{\partial x} = 1} \quad \underline{q} \cdot \underline{n} = f \\ y = 0, y = 1, \quad \cancel{\frac{\partial T}{\partial n}} = 0 \quad \underline{q} \cdot \underline{n} = 0 \end{cases}$$

set $\underline{q} = -\underline{\underline{D}} \nabla T$, $\underline{q} = (q_x, q_y)$

(b) Multiply the heat equation by a weight function w, with $w = 0$ at $x = 0$, and integrate over the whole domain Ω,

$$\int_\Omega w (-\nabla \cdot \underline{q} + s) \, d\Omega = 0.$$

Integration part by part.

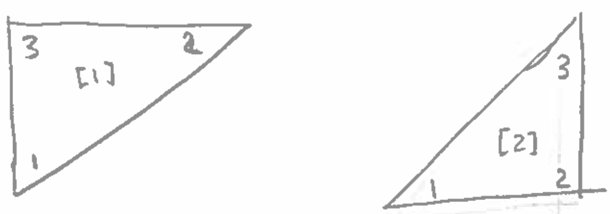
$$\int_\Omega \nabla w \cdot \underline{q} \, d\Omega - \int_{\partial\Omega} w \underline{q} \, \underline{n} \, d\Gamma + \int_\Omega w s \, d\Omega = 0$$

$$\int_\Omega (\nabla w)^T \underline{\underline{D}} \, \nabla T \, d\Omega = \int_\Omega w s \, d\Omega - \int_{\Gamma_q} w \, \overbrace{\underline{q} \cdot \underline{n}}^{f} \, d\Gamma \cancel{- \int_{\Gamma_T} \overbrace{w}^{0} \cdot \underline{q} \cdot \underline{n} \, d\Gamma}$$

Neumann Dirichlet

(c)

(i)



$A_1 = 1 \qquad \underline{\underline{B}}^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix}$

$A_2 = 1 \qquad \underline{\underline{B}}^2 = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \end{pmatrix}$

(2-2)

(ii)

$$\underline{\underline{K}}^1 = A_1 \underline{\underline{B}}^{1T} \underline{\underline{D}}\, \underline{\underline{B}}^1 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix} = \begin{array}{c|c} \begin{matrix} 1 & 3 & 4 \end{matrix} & \\ \begin{pmatrix} 4 & 0 & -4 \\ 0 & 1 & -1 \\ -4 & -1 & 5 \end{pmatrix} & \begin{matrix} 1 \\ 3 \\ 4 \end{matrix} \end{array}$$

$$\underline{\underline{K}}^2 = A_2 \underline{\underline{B}}^{2T} \underline{\underline{D}}\, \underline{\underline{B}}^1 = \begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \begin{array}{c|c} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \\ \begin{matrix} 1 & 2 & 3 \end{matrix} & \end{array}$$

(iii)

$$\int_{\Omega^e} N_I^e \, d\Omega = A_e/3 = \frac{1}{3}, \text{ for } \overset{\text{any}}{\text{shape}}\ N_I,$$

$$\underline{f}_\Omega^1 = \int_{[1]} \underline{N}^{e^T} d\Omega = \begin{array}{cc} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} & \begin{matrix} 1 \\ 3 \\ 4 \end{matrix} \end{array}, \quad \underline{f}_\Omega^2 = \int_{[2]} \underline{N}^{e^T} d\Omega = \begin{array}{cc} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

(iv) only element 2 has a non-zero contribution on the edge 2-3

$$\underline{f}_\Gamma^1 = -\underset{\substack{\| \\ 1}}{\bar{q}} \int_{\text{edge } 2\text{-}3} \underline{N}^{2^T} d\Gamma, \quad N_1^2(t) = 1-t, \quad N_1^3(t) = t$$

$$\underline{f}_\Gamma^1 = f \begin{array}{cc} \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

(v) Global matrix

$$\underline{\underline{K}} = \begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} & \\ \begin{pmatrix} 4+1 & -1 & 0 & -4 \\ -1 & 5 & -4 & 0 \\ 0 & -4 & 1+4 & -1 \\ -4 & & -1 & 5 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

$$\underline{\underline{K}}\, \underline{a} = \underline{f}$$

only need rows 2 and 3

$$\begin{pmatrix} -1 & 5 & -4 & 0 \\ 0 & -4 & 5 & -1 \end{pmatrix} \begin{pmatrix} \overset{=1}{T_1} \\ T_2 \\ T_3 \\ \underset{=1}{T_4} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{6} \end{pmatrix}$$

$$\begin{cases} 5T_2 - 4T_3 = \frac{5}{6} \\ -4T_2 + 5T_3 = \frac{7}{6} \end{cases} \qquad \begin{cases} T_2 = \frac{53}{54} \approx 0.9815 \\ T_3 = \frac{55}{54} \approx 1.0185 \end{cases}$$