Crib, 3F1 2015

1. a) Taking the $z-$transform on both sides of the difference equation (with zero initial condition) gives $zY(z) = 1/2Y(z) + U(z) + D(z)$. Rearranging yields

$$Y(z) = \frac{1}{z - 1/2}U(z) + \frac{1}{z - 1/2}D(z)$$

b) Similarly, we get $zU(z) = aU(z) - 1/4Y(z)$ or

$$U(z) = \frac{-1/4}{z - a}Y(z)$$

c) The closed-loop transfer function from $D(z)$ to $Y(z)$ is given by $Y = GU + PD = GKY + PD$. Rearranging and replacing from above gives

$$Y = \frac{P}{1 - GK}D = \frac{\frac{1}{z-1/2}}{1 + \frac{1}{z-1/2}\frac{1/4}{z-a}} = \frac{z - a}{(z - \frac{1}{2})(z - a) + \frac{1}{4}}$$

d) To start, assume that we pick a value for a that results in a stable closed-loop system (or otherwise, y_k would be unbounded). If d_k is a step, then

$$D(z) = \frac{z}{z - 1}$$

The output is given by $Y = MD$. Under the assumption that M is stable, the final value theorem gives

$$\lim_{k\to\infty} y_k = \lim_{z\to 1}(z - 1)Y(z) = \frac{1 - a}{(1 - \frac{1}{2})(1 - a) + \frac{1}{4}}$$

The effect of the disturbance is minimised in steady-state when $\lim_{k\to\infty} y_k = 0$. This can be obtained with $a = 1$. It still remains to check that the closed loop system is stable with $a = 1$. In this case, the poles of the closed-loop system are the roots of $(z - \frac{1}{2})(z - 1) + \frac{1}{4} = 0$, or $z = 0.75 \pm i0.433$. Since the poles are inside the unit disk, the closed-loop system is stable and the final value theorem applies.

e) When $a = 0$, the system is stable since the closed-loop poles are at $z = 0.25 \pm i0.433$. Hence, the output of the system as k becomes large is given by

$$y_k = |M(e^j)| \cos(k + \angle M(e^j)) = 1.5267 \cos(k - 1.2997)$$

2. a) (i) At the frequency $\omega = \pi$, $G(-1) = 1/3$. That rules out both B (-1/3) and D (2/3). Since G has two poles, the phase varies $-\pi$ per pole as ω varies from 0 to π. Hence, A cannot be. The correct answer is C.

(ii) In the log scale, the magnitude simply gets multiplied by 10. As for the phase, it also gets multiplied by 10. Hence, it was only necessary to take the frequency plot C and multiply the y axis by 10 (see Fig. 1).

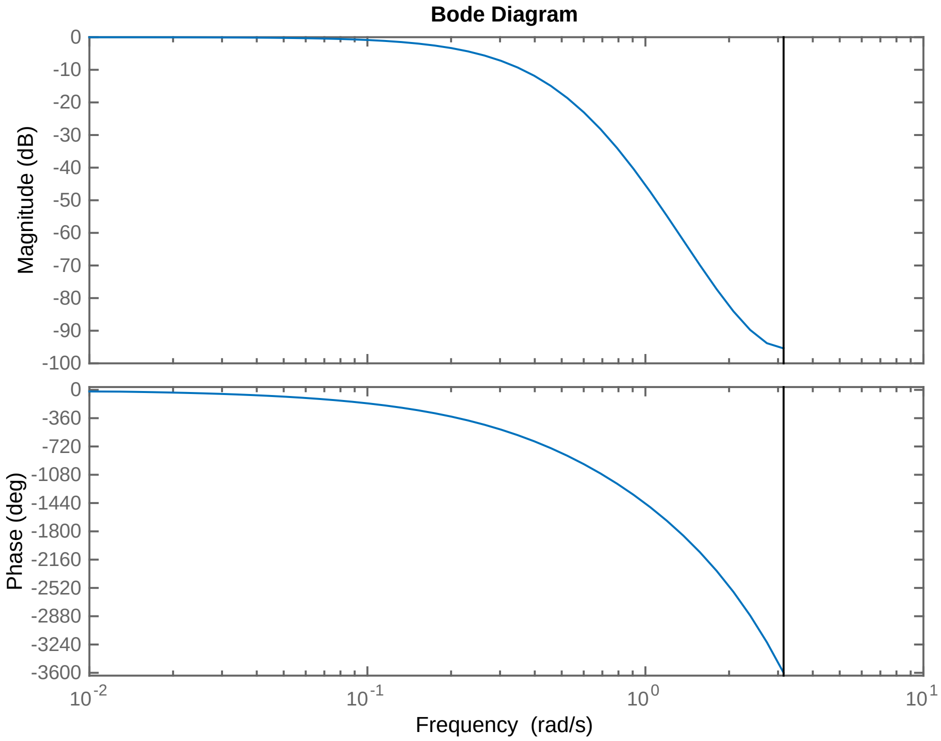

Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
Phase (deg)
0
-360
-720
-1080
-1440
-1800
-2160
-2520
-2880
-3240
-3600
10^-2
10^-1
10^0
10^1
Frequency (rad/s)

Figure 1:

b) (i)

$$
\begin{aligned}
F_Y(y) &= \Pr(Y \le y) = \Pr(g(X) \le g(x)) \\
&= \Pr(X \le x) = F_Y(x) \text{ since } g \text{ is increasing.} \\
f_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{dF_X(x)}{dy} = \\
& \frac{dF_X(x)/dx}{dy/dx} \text{ but } \frac{dy}{dx} = g'(x) \text{ hence} \\
f_Y(y) &= \frac{f_Y(x)}{g'(x)} \text{ where } y = g(x).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\frac{1}{\pi(1+y^2)} &= \frac{1/\pi}{g'(x)} \text{ where } y = g(x) \\
g'(x) &= (1 + y^2(x))
\end{aligned}
$$

Fromt the hint, $g(x) = \tan(x)$ satisfies this relationship.

3. (a) $Y = X_1 + X_2$, so if $X_1 = x_1$, then $Y = y$ if and only if $X_2 = y - x_1$.

$$f_{Y,X_1}(y,x_1) = \underbrace{f_{Y|X_1}(y|x_1)}_{f_{X_2}(y-x_1)} f_{X_1}(x_1)$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{Y|X_1}(y|x_1) f_{X_1}(x_1) dx_1 = \int_{-\infty}^{\infty} f_{X_2}(y-x_1) f_{X_1}(x_1) dx_1 \\ &= f_{X_1} \star f_{X_2} \end{aligned}$$

(b) $\Phi_X(u) = \mathrm{E}\, e^{juX} = \displaystyle\int_{-\infty}^{\infty} e^{jux} f_X(x) dx = F(-u)$ where $F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x} f_X(x) dx$ is the Fourier transform of the pdf.

$$\begin{aligned} \Phi_Y(u) &= \int_{-\infty}^{\infty} e^{-j\omega y} f_Y(y) dy = \int_{-\infty}^{\infty} e^{-j\omega y} \int_{-\infty}^{\infty} f_{X_2}(y-x_1) f_{X_1}(x_1) dx_1 dy \\ &= \int_{-\infty}^{\infty} f_{X_1}(x_1) \left[\int_{-\infty}^{\infty} f_{X_2}(y-x_2) e^{-j\omega y} dy \right] dx_1 \begin{cases} u = y - x_1 \\ y = u + x_1 \end{cases} \\ &= \int_{-\infty}^{\infty} f_{X_1}(x_1) \left[\int_{-\infty}^{\infty} f_{X_2}(u) e^{-j\omega u} du \right] e^{-j\omega x_1} dx_1 \\ &= \int_{-\infty}^{\infty} f_{X_1}(x_1) \Phi_{X_2}(u) e^{-j\omega x_1} dx_1 = \Phi_{X_1}(u) \Phi_{X_2}(u) \end{aligned}$$

(c) $Y = \displaystyle\sum_{i=1}^{N} X_i$, convolution is commutative, so

$$f(y) = f_n \star \underbrace{(f_{n-1} \star \ldots \star f_1)}_{(X_1 + \ldots + X_{n-1})} = f_n \star \ldots \star f_1$$

implying that

$$\Phi_Y(u) = \Phi_{X_n}(u) \Phi_{X_1 + \ldots + X_{n-1}}(u) = \prod_{n=1}^{N} \Phi_{X_n}(u)$$

(d)

$$\begin{aligned} \Phi_X(u) = \mathrm{E}\, e^{juX} &= (1-p) e^{ju0} + p e^{ju1} \\ &= 1 - p + p e^{ju} \end{aligned}$$

(e) <u>*Simple Way*</u>

Let X_i be the binary randon variable indicating success ($X_i = 1$) or failure ($X_i = 0$) of the i^{th} trial.

$$Y = \sum_{i=1}^{N} X_i, \Phi_{X_i}(u) = (1-p) + p e^{ju}$$

$$\text{from 3(c)} \Phi_Y u = \prod_{i=1}^{N} \Phi_{X_i}(u) = \left[(1-p) + p e^{ju} \right]^N$$

Direct Calculation

$$\begin{aligned}
\Phi_Y(u) &= \mathrm{E}\, e^{juY} \\
&= \sum_{y=0}^{N} P_Y(y) e^{juy} \\
&= \sum_{y=0}^{N} \binom{N}{y} p^y (1-p)^{N-y} e^{juy} \\
&= \sum_{y=0}^{N} \binom{N}{y} \left[p e^{ju}\right]^y (1-p)^{N-y} \text{ (Binomial series)} \\
&= \left[(1-p) + p e^{ju}\right]^N
\end{aligned}$$

4. a) Apply Kraft's inequality to check whether there exists a binary prefix-free code,

   (i) $2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} = 0.96875 < 1$ hence yes, there exists a prefix-free code.

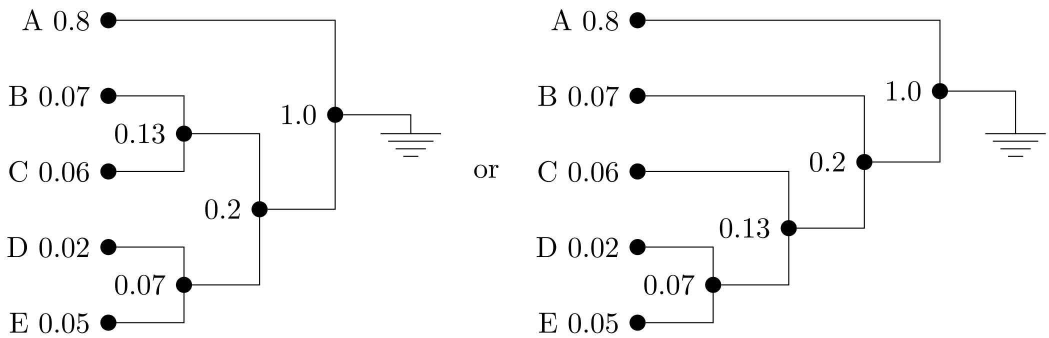
   (ii) $2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} + 2^{-4} = 1.0625 > 1$ hence no, there exists no prefix-free code.

   (iii) $5 \times 2^{-2} = 1.25 > 1$ hence no, there exists no prefix-free code.

   (iv) $2 \times 2^{-2} + 3 \times 2^{-3} = 7/8 < 1$ hence yes, there exists a prefix-free code.

b) (i) $H(X) = -0.8 \ln 0.8 - 0.07 \ln 0.07 - 0.06 \ln 0.06 - 0.02 \ln 0.02 - 0.05 \ln 0.05 = 0.76$ nats Divide by $\ln 2$ to obtain the entropy in bits, i.e., 1.10 bits.

   (ii) The Huffman algorithm has two possible outcomes due to a tie (students only expected to give one):



resulting in the codes

Symbol	Codeword		Symbol	Codeword
A	0		A	0
B	100	or	B	10
C	101		C	110
D	110		D	1110
E	111		D	1111

The average codeword length can be calculated as $E[L] = \sum_i p_i l_i$ or using the path length lemma by summing all intermediate node probabilities in the tree,

$$E[L] = 1.0 + 0.2 + 0.13 + 0.07 = 1.4$$

   (iii) For any Huffman code, we know that the expected codeword length is upper bounded as $E[L] < H(X) + 1$. For the per symbol expected codeword length when encoding blocks of symbols, we divide by the block length K to obtain

$$\frac{E[L]}{K} < \frac{H(X_1 \dots X_K)}{K} + \frac{1}{K} = H(X) + \frac{1}{K}$$

where the equality follows from the fact that $X_1, \dots, X_K$ are independent and identically distributed. Hence, the upper bounds for $K = 1, 2, 10$ are

$$E[L] < H(X) + 1 = 2.1$$
$$\frac{E[L_2]}{2} < H(X) + \frac{1}{2} = 1.6$$
$$\frac{E[L_{10}]}{10} < H(X) + \frac{1}{10} = 1.2$$

c) (i) If there was a j such that $l_j > l_{j+1}$, we could obtain a new code by switching the codewords for the j-th and the $(j+1)$-th symbol for which

$$E[L'] = \sum_{i \notin \{j,j+1\}} p_i l_i + p_j l_{j+1} + p_{j+1} l_j < \sum_i p_i l_i = E[L]$$

but this is not possible since the Huffman code is optimal in that it minimizes the average codeword length. Hence we conclude that $l_j \leq l_{j+1}$ for all $j \in \{1, 2, 3, 4\}$.

(ii) Since p_5 is smaller than p_1, p_2, p_3 and p_4, in the new distribution the last two probabilities $p_5/2$ are the smallest probabilities. The Huffman algorithm will begin by linking these two together, and the resulting intermediate node will have probability p_5, so the algorithm will continue in the same manner as it did for the 5-ary alphabet with probabilities p_1, p_2, p_3, p_4 and p_5. Hence, the lengths will satisfy $l'_i = l_i$ for $i = 1, 2, 3, 4$ and $l'_5 = l'_6 = l_5 + 1$.

Comments on Questions

1. A popular question, well answered by most candidates. The question was easy in general except for parts (d) and (e) where less than 10% of candidates remembered to check for stability when applying the final value theorem and computing a steady state response.

2. An unpopular question, partly because it mixes two different parts of the course, and partly because the digital control had a slightly a-typical question requiring candidates to understand what happens to a plot in logarithmic scale when the function is taken to the 10^{th} power. Those who did attempt the question did very well in general.

3. The most popular question, well answered by the majority of candidates. This was partly book work and partly a direct application of what had been learned during the lectures and most candidates solved it easily.

4. A less popular question that was longer than most but was solved fairly well by those who attempted it. Parts (a) and (b) were direct applications of techniques learned in the lecture, while part (c) invited further thought and only few candidates solved (c)(i) satisfactorily by referring to the optimality of Huffman's algorithm.

JS, May 2015