Crib, 3F1 2016

1. a) Any or all of these are equally acceptable:

$$y_k = g_k * u_k = \sum_{\ell=0}^{k} g_\ell u_{k-\ell} = \sum_{\ell=0}^{k} g_{k-\ell} u_\ell$$

b)

$$x_k * y_k = \sum_{\ell=0}^{k} x_\ell y_{k-\ell}$$

hence

$$\begin{aligned}
\mathcal{Z}\{x_k * y_k\} &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} x_\ell y_{k-\ell} z^{-k} && (1)\\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} x_\ell z^{-\ell} y_{k-\ell} z^{-(k-\ell)} && (2)\\
&= \sum_{\ell=0}^{\infty}\sum_{k=\ell}^{\infty} x_\ell z^{-\ell} y_{k-\ell} z^{-(k-\ell)} && (3)\\
&= \sum_{\ell=0}^{\infty} x_\ell z^{-\ell} \sum_{m=0}^{\infty} y_m z^{-m} \qquad (\text{letting } m = k-\ell) && (4)\\
&= \mathcal{Z}\{x_k\}\mathcal{Z}\{y_k\} && (5)
\end{aligned}$$

c) There are at least 2 ways of doing this, both equally good.
Method 1: The unit pulse input is $\{u_k\} = (1, 0, 0, \dots)$.
Hence from the difference equation we have (assuming $u_k = 0$ for $k < 0$ as usual)

$$\begin{aligned}
g_0 &= 0+0+0 = 0 && (6)\\
g_1 &= 1+0+0 = 1 && (7)\\
g_2 &= 0+2+0 = 2 && (8)\\
g_3 &= 0+0+1 = 1 && (9)\\
g_4 &= 0+0+0 = 0 && (10)\\
&\ \vdots && (11)
\end{aligned}$$

so that $\{g_k\} = (0, 1, 2, 1, 0, 0, \dots)$ is the pulse response.

The transfer function is the z-transform of the pulse response, so

$$G(z) = \sum_{k=0}^{\infty} g_k z^{-k} = 0 + z^{-1} + 2z^{-2} + z^{-3} + 0 + \cdots = \frac{z^2+2z+1}{z^3} = \frac{(z+1)^2}{z^3}$$

Method 2: Take z-transforms of both sides of the difference equation (assuming zero initial conditions):

$$Y(z) = z^{-1}U(z) + 2z^{-2}U(z) + z^{-3}U(z) = (z^{-1} + 2z^{-2} + z^{-3})U(z)$$

Hence the transfer function is

$$G(z) = \frac{Y(z)}{U(z)} = z^{-1} + 2z^{-2} + z^{-3} = \frac{z^2+2z+1}{z^3} = \frac{(z+1)^2}{z^3}$$

The pulse response is the inverse z-transform of the transfer function, hence

$$\{g_k\} = \mathcal{Z}^{-1}\{z^{-1} + 2z^{-2} + z^{-3}\} = (0, 1, 2, 1, 0, 0, \dots)$$

d) Again there are at least 2 ways of doing this, both equally good. Method 1 is a bit simpler in this case, but only works in some examples; method 2 is more generic.
Method 1: Directly from the difference equation, using $\{u_k\} = (u_0, u_1, u_2, \dots) = (1, -1, 1, \dots)$, gives

$$\begin{aligned}
y_0 &= 0+0+0=0 && (12)\\
y_1 &= 1+0+0=1 && (13)\\
y_2 &= -1+2+0=1 && (14)\\
y_3 &= 1-2+1=0 && (15)\\
y_4 &= -1+2-1=0 && (16)\\
&\vdots && (17)
\end{aligned}$$

and it can be seen that $y_k = 0$ for $k \geq 3$.
Method 2: From the z-transform table, $U(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1}$.
This could also be obtained from first principles as follows:

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k} = \sum_{k=0}^{\infty} (-1)^k z^{-k} = \sum_{k=0}^{\infty} \left(-\frac{1}{z}\right)^k = \frac{1}{1+1/z} = \frac{z}{z+1}$$

Therefore

$$Y(z) = G(z)U(z) = \frac{(z+1)^2}{z^3} \times \frac{z}{z+1} = \frac{z+1}{z^2} = z^{-1} + z^{-2}$$

hence $\{y_k\} = \mathcal{Z}^{-1}\{z^{-1} + z^{-2}\} = (0, 1, 1, 0, 0, \dots)$.

e) $u_k = (-1)^k = \cos(k\pi)$ so the steady-state response to this periodic input is obtained as $G(e^{j\theta})$ with $\theta = \pi$, namely as $G(-1)$. But $G(z) = (z+1)^2/z^3$, so $G(-1) = 0$, and indeed we found that $y_k = 0$ for $k \geq 3$.
(Usually the frequency response gives the response as $k \to \infty$, but in this case it gets it right after only a finite number of steps, because the given system is an FIR filter.)

2. a) (i)

$$H(z) = G\left(\frac{z-1}{T}\right) = \frac{\left(\frac{z-1}{T}\right)^2 + \omega_o^2}{\left(\frac{z-1}{T} + 10\omega_o\right)^2} \tag{18}$$

$$= \frac{(z-1)^2 + (\omega_o T)^2}{(z-1+10\omega_o T)^2} \tag{19}$$

$$= \frac{z^2 - 2z + 1 + (\omega_o T)^2}{(z-1+10\omega_o T)^2} \tag{20}$$

Hence the **poles** are both at $z = 1 - 10\omega_o T$ and
the **zeros** are at the roots of the numerator, namely at

$$\frac{1}{2}\left(2 \pm \sqrt{4 - 4[1 + (\omega_o T)^2]}\right) = 1 \pm \sqrt{-(\omega_o T)^2} = 1 \pm j\omega_o T$$

(ii) The filter is stable providing that $|1 - 10\omega_o T| < 1$. But $1 - 10\omega_o T$ is real and $\omega_o T > 0$, so this condition translates into $0 < 10\omega_o T < 2$, or

$$0 < T < \frac{1}{5\omega_o}$$

(iii) Low frequency behaviour: $s = 0$ or $z = 1$ $(= e^{j0})$:

$$G(0) = \frac{1}{100} \quad \text{and} \quad H(1) = \frac{(j\omega_o T)(-j\omega_o T)}{(10\omega_o T)^2} = \frac{1}{100}$$

so the two filters agree at frequency 0 (for all T).
High frequency behaviour: $s = \infty$ or $z = -1$ $(= e^{j\pi})$:

$$G(\infty) = 1 \quad \text{and} \quad H(-1) = \frac{1 + 2 + [1 + (\omega_o T)^2]}{(-1-1+10\omega_o T)^2} = \frac{4 + (\omega_o T)^2}{(-2+10\omega_o T)^2} \to 1 \text{ as } T \to 0$$

so the two filters agree at high frequencies as T is reduced to 0. Note that the frequency coresponding to $z = -1$ is π/T, which approaches ∞ as $T \to 0$.
Alternative solution: Low frequency: $H(1) = G\left(\frac{1-1}{T}\right) = G(0)$.
High frequency: $H(-1) = G\left(\frac{-1-1}{T}\right) = G\left(\frac{-2}{T}\right) \to G(0)$ as $T \to 0$.
So the two filters will always agree at low and high frequencies if the Euler transformation is used and $T \to 0$.

b) (i) Definition of strict sense and wide sense stationarity from lecture notes.

(ii)

$$
\begin{aligned}
EX(t) =& \int \int_{-\pi}^{\pi} f(\omega) \frac{1}{2\pi} A\cos(\omega t + \phi) d\omega d\phi \\
=& A \int \int_{-\pi}^{\pi} f(\omega) \frac{1}{2\pi} [\cos \omega t \cos \phi - \sin \omega t \sin \phi] d\omega d\phi \\
=& A \int f(\omega) \cos \omega t d\omega \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos \phi d\phi - A \int f(\omega) \sin \omega t d\omega \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin \phi d\phi \\
=& A \int f(\omega) \cos \omega t d\omega \frac{1}{2\pi} [\sin \phi]_{-\pi}^{\pi} + A \int f(\omega) \sin \omega t d\omega [\frac{1}{2\pi} \cos \phi]_{-\pi}^{\pi} \\
=& 0 \\
EX(t_1)X(t_2) =& A^2 \int \int_{-\pi}^{\pi} f(\omega) \frac{1}{2\pi} \cos(\omega t_1 + \phi) \cos(\omega t_2 + \phi) d\omega d\phi \\
=& A^2 \int \int_{-\pi}^{\pi} f(\omega) \frac{1}{2\pi} [\cos(\omega t_1 + \phi - \omega t_2 - \phi) + \cos(\omega t_2 + \phi + \omega t_1 + \phi)] d\omega d\phi \\
=& A^2 \int f(\omega) \cos \omega (t_1 - t_2) d\omega \int_{-\pi}^{\pi} \frac{1}{2\pi} d\phi \\
&+ A^2 \int \int_{-\pi}^{\pi} f(\omega) \frac{1}{2\pi} \cos(\omega(t_1 + t_2) + 2\phi) d\omega d\phi \\
=& A^2 \int f(\omega) \cos \omega (t_1 - t_2) d\omega \\
&+ \int \int_{-\pi}^{\pi} f(\omega) \frac{1}{2\pi} [\cos \omega (t_1 + t_2) \cos 2\phi - \sin \omega (t_1 + t_2) \sin 2\phi] d\omega d\phi \\
=& A^2 \int f(\omega) \cos \omega (t_1 - t_2) d\omega + A^2 \int f(\omega) \cos \omega (t_1 + t_2) d\omega \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos 2\phi d\phi \\
&- A^2 \int \sin f(\omega) \omega (t_1 + t_2) d\omega \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin 2\phi d\phi \\
=& A^2 \int f(\omega) \cos \omega (t_1 - t_2) d\omega
\end{aligned}
$$

$EX(t_1)X(t_2)$ is a function of $|t_1 - t_2|$ and $EX(t)$ is constant, so $X(t)$ is WSS.

3. (a) Definition of ergodic from lecture notes.

   (b) With $EX(t) = 0$, $r_{XX}(0) = EX(t)X(t) = EX(t)^2$ for any t, so that $Q_X = EX(t)^2$. From the pdf of $X(t)$, $f_X(x) = \frac{1}{2A}$ for $x \in [-A, A]$ so that

$$EX(t)^2 = \int_{-A}^{A} x^2 \frac{1}{2A} dx = \frac{1}{2A} \frac{x^3}{3} |_{-A}^{A} = \frac{A^2}{3}$$

   therefore $Q_X = \frac{A^2}{3}$.

   (c)

$$\begin{aligned} S_X(\omega) =& Q_X \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} e^{-j\omega\tau} d\tau \\ =& Q_X \int_{-\infty}^{0} e^{2\lambda\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\lambda\tau} e^{-j\omega\tau} d\tau \\ =& \frac{Q_X}{2\lambda - j\omega} e^{2\lambda\tau - j\omega\tau} |_{-\infty}^{0} - \frac{Q_X}{2\lambda + j\omega} e^{2\lambda\tau - j\omega\tau} |_{0}^{\infty} + \\ =& \frac{Q_X}{2\lambda - j\omega} + \frac{Q_X}{2\lambda + j\omega} = Q_x \frac{4\lambda}{4\lambda^2 + \omega^2} \end{aligned}$$

   (d)

$$\begin{aligned} Y(t) =& X(t) * h(t) = X(t) * \delta(t) + X(t) * \delta(t - T) = x(t) - x(t - T) \\ r_{YY}(\tau) =& E[Y(t)Y(t - \tau)] = E[(X(t) - X(t - T))(X(t - \tau) - X(t - T - \tau)] \\ =& EX(t)X(t - \tau) - EX(t)X(t - T - \tau) - EX(t - T)X(t - T - \tau) + EX(t - T)X(t - T - \tau \\ =& r_{XX}(\tau) - r_{XX}(\tau - T) - r_{XX}(\tau + T) + r_{XX}(\tau) \\ =& 2r_{XX}(\tau) - r_{XX}(\tau - T) - r_{XX}(\tau + T) \end{aligned}$$

   $S_Y(\omega)$ is the Fourier Transform of $r_{YY}(\tau)$, and $S_X(\omega)$ is the Fourier Transform of $r_{XX}(\tau)$.

$$\begin{aligned} S_Y(\omega) =& 2S_X(\omega) - e^{-j\omega T} S_X(\omega) - e^{j\omega T} S_X(\omega) \\ =& S_X(\omega)(2 - (e^{-j\omega T} + e^{j\omega T})) = 2S_X(\omega)(1 - \cos\omega T) \\ =& 2Q_X \frac{4\lambda}{4\lambda^2 + \omega^2}(1 - \cos\omega T) \end{aligned}$$

4. a) X is uniformly distributed over its ternary alphabet $\{1, 2, 3\}$ hence

$$H(X) = \log_2 3 = 1.585 \text{ [bits]}$$

b) The number of female offspring follows a binomial distributed given the total number of offspring, i.e.,

$$\begin{cases} P_{Y|X}(0|1) = 1/2 \\ P_{Y|X}(1|1) = 1/2 \end{cases}$$

and

$$\begin{cases} P_{Y|X}(0|2) = 1/4 & \text{(all male)} \\ P_{Y|X}(1|2) = 1/2 & \text{(male,female) or (female,male)} \\ P_{Y|X}(2|2) = 1/4 & \text{(all female)} \end{cases}$$

and

$$\begin{cases} P_{Y|X}(0|3) = 1/8 & \text{(all male)} \\ P_{Y|X}(1|3) = 3/8 & \text{(m,m,f) or (m,f,m) or (f,m,m)} \\ P_{Y|X}(2|3) = 3/8 & \text{(m,f,f) or (f,m,f) or (f,f,m)} \\ P_{Y|X}(3|3) = 1/8 & \text{(all female)} \end{cases}$$

hence

$$\begin{aligned} H(Y|X) &= H(Y|X = 1)P_X(1) + H(Y|X = 2)P_X(2) + H(Y|X = 3)P_X(3) \\ &= H(1/2, 1/2)\frac{1}{3} + H(1/4, 1/2, 1/4)\frac{1}{3} + H(1/8, 3/8, 3/8, 1/8)\frac{1}{3} \\ &= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\log(2) + \frac{1}{2}\log(4)\right) + \frac{1}{3}\left(\frac{1}{4}\log(8) + \frac{3}{4}\log\frac{8}{3}\right) \\ &= \frac{1}{3} + \frac{1}{3}\frac{3}{2} + \frac{1}{3}\left(3 - \frac{3}{4}\log(3)\right) \\ &= \frac{11}{6} - \frac{1}{4}\log(3) = 1.437 \text{ [bits]} \end{aligned}$$

c)

$$\begin{aligned} H(X, Y) &= H(X) + H(Y|X) = \log(3) + \frac{11}{6} - \frac{1}{4}\log(3) = \frac{3}{4}\log(3) + \frac{11}{6} \\ &= 1.585 + 1.437 = 3.022 \text{ [bits]} \end{aligned}$$

d) We compute the joint probabilities $P_{XY}(x, y) = P_X(x)P_{Y|X}(y|x)$ and construct the code tree using Huffman's algorithm as shown in Figure 1.
The resulting code table is

(X, Y)	Codeword
$(1, 0)$	000
$(1, 1)$	001
$(2, 0)$	100
$(2, 1)$	010
$(2, 2)$	101
$(3, 0)$	1100
$(3, 1)$	011
$(3, 2)$	111
$(3, 3)$	1101

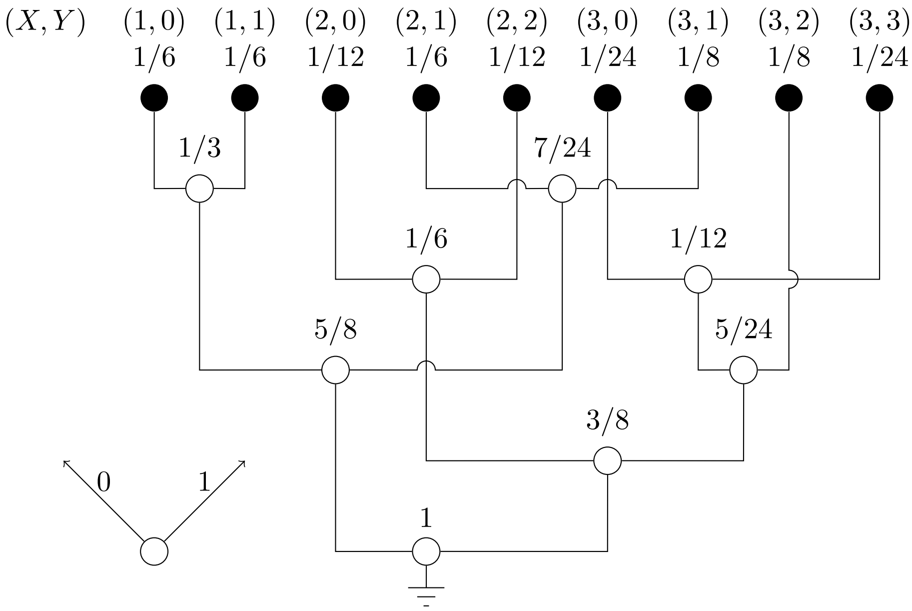

Figure 1: Huffman tree for Question 4.d)

e)

$$E[L] = \frac{1}{6}3 + \frac{1}{6}3 + \frac{1}{12}3 + \frac{1}{6}3 + \frac{1}{12}3 + \frac{1}{24}4 + \frac{1}{8}3 + \frac{1}{8}3 + \frac{1}{24}4$$
$$= 3(1 - 1/12) + 4/12 = 3\ 1/12 = 3.0833$$

and hence

$$\kappa_1 = E[L]/L_f = 3.0833/4 = 0.7708.$$

f)

$$E[L_{SF}] = \sum_{(} x, y)P_{XY}(x, y) \left\lceil \log \frac{1}{P_{XY}(x, y)} \right\rceil$$
$$= 3\left(3\frac{1}{6} + 2\frac{1}{8}\right) + 4\left(2\frac{1}{12}\right) + 5\left(2\frac{1}{44}\right)$$
$$= \frac{9}{4} + \frac{2}{3} + \frac{5}{12} = 10/3 = 3.3333$$

and hence

$$\kappa_2 = E[L_{SF}]/L_H = 3.3333/4 = 5/6 = 0.8333$$

g) When encoding database entries jointly using an arithmetic encoder, the overall codeword length will tend to the block entropy plus 2, and hence the entropy per database entry will tend to the entropy since the effect of the added 2 divided by the number of database entries will vanish. Hence we have

$$\kappa_3 = H(X, Y)/L_f = 3.0221/4 = 0.7555.$$

Comments on Questions

1. To be completed...
2. To be completed...
3. To be completed...
4. To be completed...

JS, January 2016