Question 1. *The question was attempted by all but one student. Most students demonstrated grasp of the z-transform, recall definitions and were able to perform appropriate manipulations to find transfer functions and time-domain behaviour. The main mistakes were forgetting technical conditions for stability and boundedness (perhaps by rushing) and various slips in algebra.*

(a)(i) System is *linear* and *time invariant* $\implies$ transfer function exists.
System *stable* $\iff$ (bounded $u \implies$ bounded y).

(a)(ii) The pulse response is $\{a_0, a_1, ...a_N\}$, finite. The system is stable because: pulse response finite, therefore absolutely bounded OR all poles are at the origin.

(b) Z-transform gives:

$$z\bar{y} - z\alpha + a\bar{y} = \bar{u} - bz^{-1}\bar{u}$$

(since $u_{-1} = 0$). Putting $u_k = \{1\}_{k\geq 0}$, we have $\bar{u} = \frac{1}{1-z^{-1}}$. Substituting and rearranging:

$$z\bar{y} - z\alpha + a\bar{y} = (1 - bz^{-1})\frac{1}{1 - z^{-1}}$$

$$\bar{y}(z + a) = (1 - bz^{-1})\frac{1}{1 - z^{-1}} + z\alpha$$

$$\bar{y} = (1 - bz^{-1})\frac{1}{(z + a)(1 - z^{-1})} + \frac{z\alpha}{(z + a)}$$

$$\bar{y} = z^{-1}(1 - bz^{-1})\frac{1}{(1 + az^{-1})(1 - z^{-1})} + \frac{\alpha}{(1 + az^{-1})}$$

Decompose quotient on right-hand side into partial fractions:

$$\frac{1}{(1 + az^{-1})(1 - z^{-1})} = \frac{A}{(1 + az^{-1})} + \frac{B}{(1 - z^{-1})}, \text{where } A = (1-a^{-1})^{-1}, B = (1+a)^{-1}$$

Then

$$\bar{y} = z^{-1}\left(\frac{A}{(1 + az^{-1})} + \frac{B}{(1 - z^{-1})}\right) - bz^{-2}\left(\frac{A}{(1 + az^{-1})} - \frac{B}{(1 - z^{-1})}\right) + \frac{\alpha}{(1 + az^{-1})}$$

So

$$y_k = \{B + A(-a)^{k-1}\}_{k\geq 1} + b\{B + A(-a)^{k-2}\}_{k\geq 2} + \alpha\{(-a)^k\}_{k\geq 0}$$

(c)(i) Using part (b) with $\alpha = 0$, transfer function is:

$$\frac{\bar{y}}{\bar{u}} = \frac{(1 - bz^{-1})}{z + a} = z^{-1}\frac{(z - b)}{z + a}$$

System is stable provided $|a| < 1$.

(c)(ii) Let $u_k = -cy_k + r_k$ and substitute into difference equation with $\alpha = 0$:

$$y_{k+1} + ay_k = -cy_k - bcy_{k-1} - r_k - br_{k-1}$$

Take z-transform and rearrange:

$$z\bar{y} + (a + c)\bar{y} + z^{-1}bc\bar{y} = -\bar{r} - z^{-1}b\bar{r}$$

The transfer function is:

$$\frac{\bar{y}}{\bar{r}} = -\frac{(1 - bz^{-1})}{z + (a + c) + z^{-1}bc} = -\frac{(z - b)}{z^2 + (a + c)z + bc}$$

For the case where $a = b$, the transfer function is:

$$\frac{\bar{y}}{\bar{r}} = -\frac{(z - b)}{z^2 + (b + c)z + bc} = -\frac{(z - b)}{(z + b)(z + c)}$$

The new system is stable provided $|b|, |c| < 1$. The new system is causal.

Question 2. *Popular question, with high marks in general. The theoretical material in (a), (b) was well received. Very few mistakes in filter design in (c), notably on the computation of the normalised cutoff frequency and on the derivation of the difference equation. (d) was straightforward. (e) was well answered in general but the discussion of the phenomenon of aliasing was sometime poor or students did not apply Shannon-Nyquist theorem properly.*

(a) The bilinear transform maps stable filters into stable filters therefore $F_d(z)$ is stable. Every stable pole of the analog prototype is mapped into a pole whose radius is less than or equal to one. See Figure 1.

The mapping is explained by the following mathematical argument. Given the bilinear transform $s = \frac{z-1}{z+1}$ solve for z to get $z = \dfrac{1+s}{1-s}$. The generic pole $s = \lambda + j\omega$ of the analog prototype is mapped into the pole z of the digital filter, whose magnitude satisfies

$$|z|^2 = zz^* = \frac{1 + (\lambda + j\omega)}{1 - (\lambda + j\omega)} \cdot \frac{1 + (\lambda - j\omega)}{1 - (\lambda - j\omega)} = \frac{(1 + \lambda)^2 + \omega^2}{(1 - \lambda)^2 + \omega^2}$$

For $\lambda \leq 0$ (stable pole) we have $(1 + \lambda)^2 \leq (1 - \lambda)^2$. Thus,

$$|z|^2 = \frac{(1 + \lambda)^2 + \omega^2}{(1 - \lambda)^2 + \omega^2} \leq 1$$

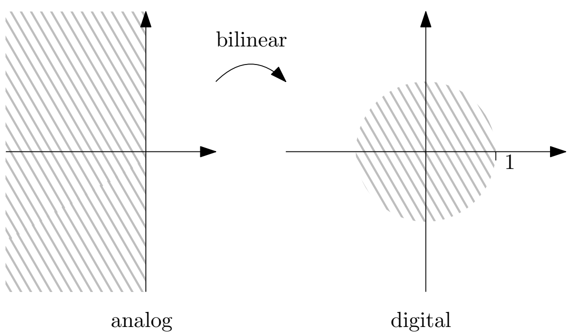


Figure 1: Every stable pole is mapped into the circle of radius one.

(b) The frequency response of the digital filter $F_d(e^{j\Omega})$ at frequency Ω corresponds to the frequency response of the analogue prototype $F_d(e^{j\Omega}) = F_a(\psi(e^{j\Omega}))$ where

$\psi(z) = \frac{z-1}{z+1}$ (bilinear transform). Thus, $F_d(e^{j\Omega}) = F_a(j\omega)$ for $j\omega = \psi(e^{j\Omega})$. It follows that

$$j\omega = \psi(e^{j\Omega}) = \frac{e^{j\Omega} - 1}{e^{j\Omega} + 1} = \frac{e^{j\Omega/2} - e^{-j\Omega/2}}{e^{j\Omega/2} + e^{-j\Omega/2}} = \frac{j\sin(\Omega/2)}{\cos(\Omega/2)} = j\tan(\Omega/2)$$

which explains the frequency warping $\omega = \tan(\Omega/2)$.

(c) The normalized cutoff frequency $\Omega_c = 0.5\pi$ is obtained by setting the analog cutoff frequency at $\omega_c = \tan(\Omega_c/2) = \tan(\pi/4) = 1$ From the analog lowpass filter, using the lowpass to highpass transformation, we get

$$F_a(\bar{s}) = \frac{\bar{s}}{\bar{s} + \omega_c} = \frac{\bar{s}}{\bar{s} + 1} \ .$$

Thus,

$$F_d(z) = F_a\left(\frac{z-1}{z+1}\right) = \frac{\frac{z-1}{z+1}}{\frac{z-1}{z+1} + 1} = \frac{z-1}{2z} = \frac{1}{2}\left(1 - z^{-1}\right) \ .$$

The difference equations that implements the filter is obtained by antitransform

$$Y(z) = F_d(z)U(z) = \frac{1}{2}(1 - z^{-1})U(z) \quad \overset{\mathcal{Z}^{-1}}{\Rightarrow} \quad y_k = \frac{1}{2}\left(u_k - u_{k-1}\right)$$

For illustration purposes, Bode diagrams of F_a and F_d are reported in Figure 2. Note that at $\omega_c = 1$ we have $|F_a(j\omega_c)| = -3\text{dB}$, as desired, which corresponds to the magnitude $|F_d(e^{j\Omega_c})| = -3\text{dB}$ at $\Omega_c = 0.5\pi$, as desired.

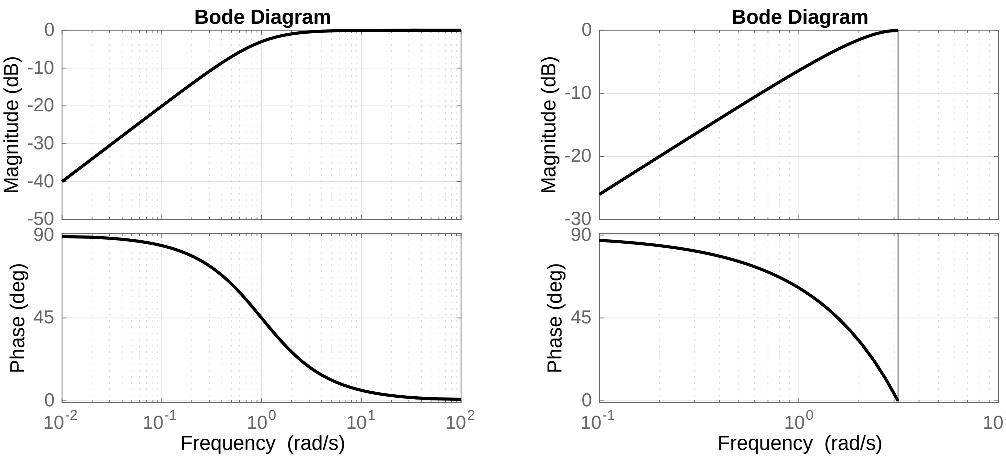


Figure 2: Left - analog prototype , Right - digital filter F_d.

(d) Sampling period $T = 0.01$ allows for a max frequency bandwidth of $\omega_{max} = \frac{2\pi}{2T} = 100\pi$ rad/s. Thus, for a cutoff frequency at 5 Hz, i.e. 10π rad/s, we need a normalized cutoff frequency at $\Omega_c = \frac{10}{100}\pi = 0.1\pi$ rad/s . For $T = 0.1$, we get $\omega_{max} = \frac{2\pi}{0.2} = 10\pi$ rad/s, and normalized cutoff frequency $\Omega_c = \pi$ rad/s .

(e) By Shannon theorem, for a maximum frequency of $f_{max} = 20$ Hz, we need $2f_{max} < \frac{1}{T}$, that is, $T < \frac{1}{2f_{max}} = \frac{1}{40} = 0.025s$.

The sampled signal spectrum is periodic with period $\omega_0 = \frac{2\pi}{T}$ rad/s, as represented in Figure 3 (left) where ω_0 is the sampling frequency. Bandlimited signals with max frequency ω_{max} can be exactly reconstructed from their samples if $2\omega_{max} < \omega_0$ (using a low pass filter) since this condition prevents overlap of the periodic signal spectrum. If $2\omega_{max} < \omega_0$ is not satisfied, distortion on the spectrum may occur, as shown in Figure 3 (right). This phenomenon is called aliasing. In such a case, the original signal cannot be reconstructed since the original high frequencies are lost.

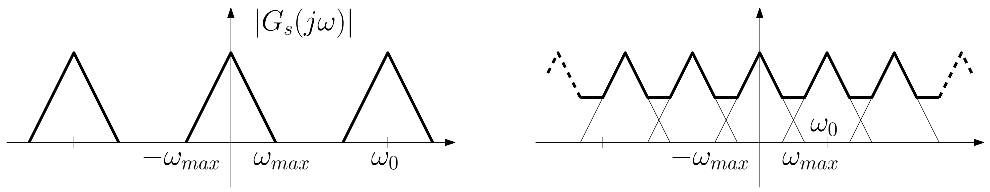


Figure 3: Left - Periodicity, no aliasing. Right - effect of aliasing on the spectrum.

Question 3. *The question was attempted by a large number of students. Simple errors with complex numbers manipulation in (b). (c) and (d) were well answered with some students using the wrong frequency on the Bode diagram. Many issues with (e), either in the selection of the Nyquist diagram or in the application of the Nyquist criterion. Most students provided the right answer to (f) but explanations were incomplete.*

(a) Directly from the expression

$$Y(z) = \gamma\left(1 + z^{-1} + z^{-2}\right)U(z) \quad \Rightarrow \quad G(z) = \gamma\left(1 + z^{-1} + z^{-2}\right)$$

(b) Method 1: compute the impulse response: $g_0 = \gamma$, $g_1 = \gamma$, $g_2 = \gamma$, $g_k = 0$ for $k \geq 3$. Thus

$$\bar{g}_p = \sum_{k=0}^{3} g_k e^{-j\frac{2\pi k}{4}p} = \gamma\left(e^{-j\frac{2\pi 0}{4}p} + e^{-j\frac{2\pi 1}{4}p} + e^{-j\frac{2\pi 2}{4}p}\right) = \gamma\left(1 + e^{-j\frac{\pi}{2}p} + (-1)^p\right) .$$

Method 2: note that

$$\bar{g}_p = \sum_{k=0}^{3} g_k e^{-j\frac{\pi k}{2}p} = \sum_{k=0}^{3} g_k z^{-k}$$

for $z = e^{j\frac{\pi}{2}p}$. The FIR filter satisfies $g_k = 0$ for $k \geq 3$ therefore

$$\sum_{k=0}^{3} g_k z^{-k} = \sum_{k=0}^{\infty} g_k z^{-k} = G(z)$$

for $z = e^{j\frac{\pi}{2}p}$. Thus,

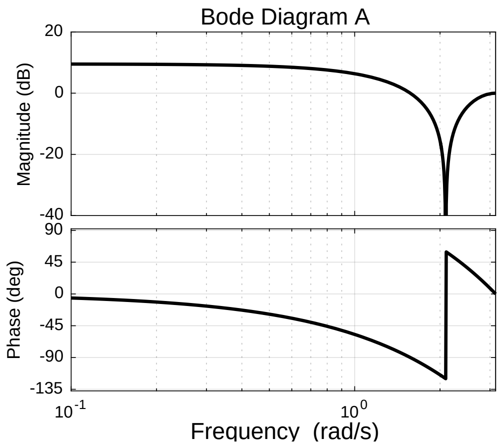
$$\bar{g}_p = G(e^{j\frac{\pi}{2}p}) = \gamma\left(1 + e^{-j\frac{\pi}{2}p} + (-1)^p\right)$$

(c) For $\gamma = 1$ we have $G(z) = 1 + z^{-1} + z^{-2} = \dfrac{z^2 + z + 1}{z^2}$.

Bode Diagram C must be excluded since $|G(e^{j0})| = |G(1)| = 3$ ($\simeq 9.54$ dB) which is not compatible with Bode Diagram C at frequency $\theta = 0$.

Bode Diagram B must be excluded since $G(z)$ has two complex conjugate zeros at $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, whose magnitude $|z| = 1$ (on the unit circle). Zeros on the unit circle guarantee that $|G(e^{j\theta})| = 0$ ($=-\infty$ dB) for some $0 \leq \theta \leq \pi$.

Bode Diagram A is the Bode diagram of $G(z)$.



(d) By the final value theorem, using linearity, the steady state response for $u_k = 2$ is

$$y_{ss}(k) = 2G(1) = 6\gamma \ .$$

Note that the steady state can be estimated directly from the Bode diagram, without any computation: $\angle G(e^{j0}) = 0$; $|G(e^{j0})| \simeq 10 + 20\log_{10}(\gamma)$ dB is an acceptable approximation. Thus, $y_{ss}(k) \simeq 2 \cdot \gamma \cdot 10^{\frac{10}{20}} \simeq 6.32\gamma$.

For $u_k = \cos(2k)$ the steady state output

$$y_{ss}(k) = |G(e^{2j})| \cos(2k + \angle G(e^{2j})) \ .$$

From the Bode diagram, $|G(e^{2j})| \simeq -16 + 20\log_{10}(\gamma)$ dB $= \gamma \cdot 10^{\frac{-16}{20}} \simeq 0.16\gamma$ and $\angle G(e^{2j}) \simeq -0.64\pi$.

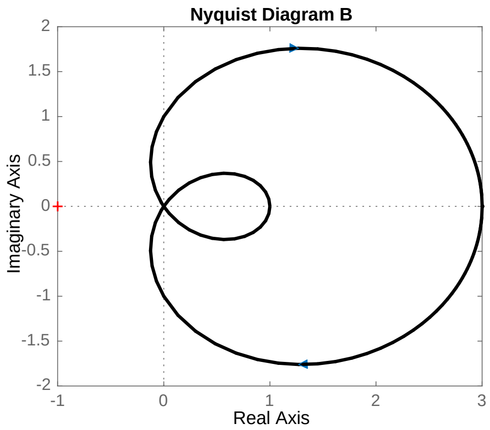
(e) Diagram B is the correct one.

From the Bode diagram, $\angle G(e^{j0}) = 0$ but the Nyquist locus of Diagram A has initial phase $-\frac{\pi}{2}$.

From the Bode diagram, $|G(e^{j\theta})| = -\infty$ dB for some θ but the Nyquist locus of Diagram C never goes to 0.

The open loop is asymptotically stable. Therefore, by Nyquist criterion, the closed loop is stable if the Nyquist locus does not encircle the point -1. Thus,

- the closed loop is stable for any $\gamma > 0$ (no encirclements)
- For $\gamma < 0$, we study the Nyquist locus of $-G(z)$, which corresponds to the Nyquist locus of $G(z)$ rotated by 180 degrees. In such a case, the closed loop is stable if $-\frac{1}{3} < \gamma \leq 0$ (no encirclements). For $\gamma \leq -\frac{1}{3}$ the number of encirclements is not defined or larger than 0. Thus the closed loop is not asymptotically stable.



(f) The closed loop transfer function from r to y is given by

$$W(z) = \frac{G(z)}{1+G(z)} = \frac{\frac{z^2+z+1}{z^2}}{1+\frac{z^2+z+1}{z^2}} = \frac{z^2+z+1}{2z^2+z+1} \, .$$

The transfer function has complex conjugated poles in $z = -\frac{1}{4} \pm \frac{\sqrt{7}}{4} i$. Poles are not all at zero thus the filter is not FIR.

Question 4. *The question was attempted by half of the students. The marks were very bimodal in the sense that if the student knew the material they were able to get a good mark and complete most of the question. Most were able to recall basic definitions but around one third were unable to apply the concepts, or forgot important relationships (such as Weiner Khinchin theorem) and did poorly.*

(a)(i) $r_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int\int x_1 x_2 f(x_1, x_2) dx_1 dx_2$, where x_1 and x_2 are the values of the random process sampled at times t_1, t_2 respectively, f is the joint PDF for x_1 and x_2, and E is expectation (integral not necessary). A process is WSS if the mean is independent of time and autocorrelation function depends only on $\tau = t_2 - t_1$.

(a)(ii) A WSS process is mean and correlation ergodic if mean and correlation functions computed over ensembles are equal to the respective values computed by averaging over time, i.e:

$$E[X] = \langle X(t) \rangle_T = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t) dt$$

$$r_{XX}(\tau) = \langle X(t)X(t+\tau) \rangle_T = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau) dt$$

(b)(i) Defining properties of white noise process: zero mean; autocorrelation function equal to delta function.

(b)(ii) First compute the response function of the system. Taking Fourier transform of the ODE:

$$j\omega C\bar{V} + (1/R)\bar{V} = \bar{I}$$

(where I is input current). Thus:

$$\mathcal{H}(\omega) = \frac{1}{j\omega C + 1/R} = \frac{R}{j\omega RC + 1}$$

and so

$$|\mathcal{H}(\omega)|^2 = \frac{R^2}{(\omega RC)^2 + 1}$$

Using the formula derived in lectures the PSD of V is:

$$\mathcal{S}_V = |\mathcal{H}(\omega)|^2 \mathcal{S}_\epsilon = |\mathcal{H}(\omega)|^2 P_0 = \frac{P_0 R^2}{(\omega RC)^2 + 1}$$

(b)(iii) Recall from lecture 15:

$$r_{VV}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{S}_V(\omega) d\omega$$

(obtained by computing inverse Fourier transform in definition of PSD.) Therefore,

$$r_{VV}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{P_0 R^2}{(\omega RC)^2 + 1} d\omega = \frac{P_0}{2\pi} \int_{-\infty}^{\infty} \frac{R^2}{(RC)^2(\omega^2 + 1/(RC)^2)} d\omega$$

$$= \frac{P_0}{2\pi C^2} \Big[RC \tan^{-1}(RC\omega) \Big]_{-\infty}^{\infty} = \frac{RP_0}{2\pi C} \Big[\pi/2 + \pi/2 \Big]$$

$$= \frac{RP_0}{2C}$$