3F1 Crib

Question 1

a) Bookwork.

b) i) No. For BIBO stability all poles must lie strictly inside unit circle (i.e. $|p|<1$). But open loop system has transfer function:

$$B(z).A(z) = \frac{6(1-3z)}{3(1-z)(1-6z)}$$

which has a pole at $z=1$.

ii) Let u_k be unit pulse. Then $\mathcal{Z}\{u\} = 1$.
So pulse response is $\mathcal{Z}^{-1}\{B(z)A(z)\}$.
Compute using partial fractions:

$$B(z)A(z) = \frac{2(1-3z)}{(1-z)(1-6z)} = \frac{P}{1-z} + \frac{Q}{1-6z}$$

$$2-6z = P(1-6z) + Q(1-z)$$

$$z=1 \Rightarrow \quad -4 = -5P \quad \therefore P = 4/5$$

$$z=1/6 \Rightarrow \quad 1 = \tfrac{5}{6}Q \quad \therefore Q = 6/5$$

$$\therefore \quad B(z).A(z) = \frac{4}{5(1-z)} + \frac{6}{5(1-6z)}$$

$$= \frac{4z^{-1}}{5(z^{-1}-1)} + \frac{6z^{-1}}{5(z^{-1}-6)}$$

$$= -\left(\frac{4}{5}\right)\underbrace{z^{-1}}_{\text{delay}}\cdot\underbrace{\left(\frac{1}{1-z^{-1}}\right)}_{\text{step}} - \left(\frac{\cancel{6}}{5}\right)\underbrace{z^{-1}}_{\text{delay}}\cdot\underbrace{\frac{1}{\cancel{6}\left(1-\frac{z^{-1}}{6}\right)}}_{\{1/6^k\}}$$

$\therefore$ pulse response $y_k = \begin{cases} 0 & k=0 \\ -\frac{4}{5} - \frac{1}{5}\cdot\frac{1}{6^{k-1}} & k \geq 1 \end{cases}$

iii) F.V.T. $\Rightarrow \lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z)Y(z)$

for system in (ii)

F.V.T. gives $\lim_{k\to\infty} y(k) = \lim_{z\to 1} \frac{2(1-3z)}{(1-6z)}$

$= -4/5$

This agrees with $\lim_{k\to\infty}$ of pulse response.

BIBO stability requires bounded output for <u>all</u> bounded input $\therefore$ system not BIBO is consistent with some bounded input having bounded output.

(iv) ~~Open~~ Closed loop transfer fn $T_c = \dfrac{BA}{1-BA}$

$$1-BA = \frac{(1-z)(1-6z) - 2(1-3z)}{(1-z)(1-6z)} = \frac{6z^2 - z - 1}{(1-z)(1-6z)}$$

$$= \frac{6\left(z^2 - \frac{z}{6} - \frac{1}{6}\right)}{(1-z)(1-6z)} = \frac{6(z+\frac{1}{3})(z-\frac{1}{2})}{(1-z)(1-6z)}.$$

$$\therefore \quad \frac{1}{1-BA} = \frac{(1-z)(1-6z)}{6(z+\frac{1}{3})(z-\frac{1}{2})}$$

$$\text{So} \quad T_c = \frac{\cancel{6}(1-3z)\cancel{(1-z)}\cancel{(1-6z)}}{3\cancel{(1-z)}\cancel{(1-6z)}\cancel{6}(z+\frac{1}{3})(z-\frac{1}{2})}$$

$$\boxed{= \frac{1-3z}{3(z+\frac{1}{3})(z-\frac{1}{2})}}$$

Stable since poles at $-\frac{1}{3}$ and $+\frac{1}{2}$ are $|\cdot|<1$

Pulse response - use partial fractions

$$T_c = \frac{\frac{1}{3} - z}{(z+\frac{1}{3})(z-\frac{1}{2})} = \frac{P}{(z+\frac{1}{3})} + \frac{Q}{(z-\frac{1}{2})}$$

$$\tfrac{1}{3} - z = P(z - \tfrac{1}{2}) + Q(z + \tfrac{1}{3})$$

$z = \frac{1}{2} \Rightarrow \quad \frac{1}{6} = \frac{5}{6}Q \quad \therefore Q = \frac{1}{5}$

$z = -\frac{1}{3} \Rightarrow \quad \frac{2}{3} = -\frac{5}{6}P \quad \therefore P = -\frac{4}{5}$

So $T_c = \dfrac{1}{5(z-1/2)} - \dfrac{4}{5(z+1/3)}$

$$= \frac{z^{-1}}{5(1-\frac{1}{2}z^{-1})} - \frac{4}{5}\cdot\frac{z^{-1}}{(1+\frac{1}{3}z^{-1})}$$

$$= \frac{1}{5}\cdot\underbrace{z^{-1}}_{\text{delay}}\cdot\underbrace{\frac{1}{(1-\frac{z^{-1}}{2})}}_{\frac{1}{2}^{k}} - \frac{4}{5}\cdot\underbrace{z^{-1}}_{\text{delay}}\cdot\underbrace{\frac{1}{(1+\frac{z^{-1}}{3})}}_{(-\frac{1}{3})^{k}}$$

So pulse response is

$$y_k = \begin{cases} 0 & k = 0 \\ \left(\frac{1}{5}\right)\left(\frac{1}{2^{k-1}}\right) - \left(\frac{4}{5}\right)\frac{1}{(-3)^{k-1}} & k \geq 1 \end{cases}$$

Question 2

(a) $\mathcal{H}$ has frequency response

$$H(e^{j\theta}) = \begin{cases} 1 & |\theta| \leq \pi/2 \\ 0 & |\theta| > \pi/2 \end{cases}$$

By inverse Fourier transform $\mathcal{F}^{-1}$

$$h_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})e^{j\theta k}d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\theta k}d\theta = \frac{\sin(k\pi/2)}{\pi k} \qquad -\infty < k < \infty$$

The filter is not realizable because it is not causal.

(b) The impulse response g_k is given by

$$g_k = h_{k-1}w_k$$

where $h_{-1} = h_1 = 1/\pi$, $h_0 = 1/2$ and $w_0 = w_2 = 0.08$, $w_1 = 1$. Thus,

$$g_0 = 0.08/\pi\ , \quad g_1 = 1/2\ , \quad g_2 = 0.08/\pi$$

and

$$y_k = \frac{0.08}{\pi}u_k + \frac{1}{2}u_{k-1} + \frac{0.08}{\pi}u_{k-2}$$

(c) In comparison to the rectangular window, the Hamming window reduces the ripples of the final filter. The product between the ideal filter and the window corresponds to convolution in frequency (duality)

$$G(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})W(e^{j(\omega-\theta)})d\theta\ .$$

The convolution between H and W introduces distortion on the ideal filter frequency response.

(d) From the difference equation the transfer function reads

$$G(z) = \frac{0.08}{\pi} + \frac{1}{2}z^{-1} + \frac{0.08}{\pi}z^{-2}$$

We compute

$$\begin{aligned} G(e^{j\theta}) &= \frac{0.08}{\pi} + \frac{1}{2}e^{-j\theta} + \frac{0.08}{\pi}e^{-2j\theta} \\ &= e^{-j\theta}\left(\frac{0.08}{\pi}e^{j\theta} + \frac{1}{2} + \frac{0.08}{\pi}e^{-j\theta}\right) \\ &= e^{-j\theta}\left(\frac{1}{2} + \frac{0.16}{\pi}\cos(\theta)\right)\ , \end{aligned}$$

which gives linear phase $-\theta$ and magnitude $\frac{1}{2} + \frac{0.16}{\pi}\cos(\theta) \simeq 0.5 + 0.05\cos(\theta)$. The magnitude plot starts at 0.55 and decreases monotonically to 0.45.

(e.i) We have

$$G(e^{j2/3\pi}) = e^{-j2/3\pi}\left(\frac{1}{2} - \frac{0.08}{\pi}\right) \simeq 0.47e^{-j2/3\pi} \ .$$

Thus, the steady state response is

$$y_k = 2|G(e^{j2/3})| \sin\left(\frac{2\pi}{3}k + \arg(G(e^{j2/3}))\right) \simeq 0.94 \sin\left(\frac{2\pi}{3}(k-1)\right) \qquad k \geq 0$$

(e.ii) We have

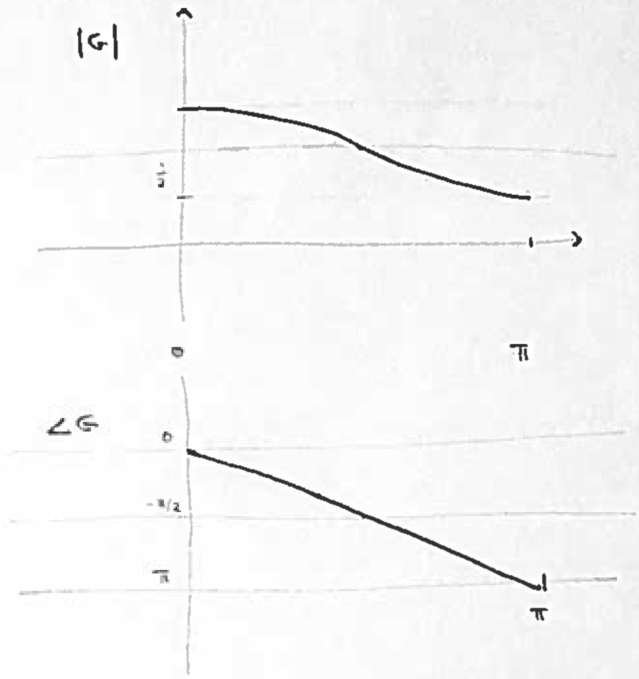
$$G(e^{j0}) = G(1) = \frac{1}{2} + \frac{0.16}{\pi} \simeq 0.55 \ .$$

For a step input of magnitude one the steady state response is

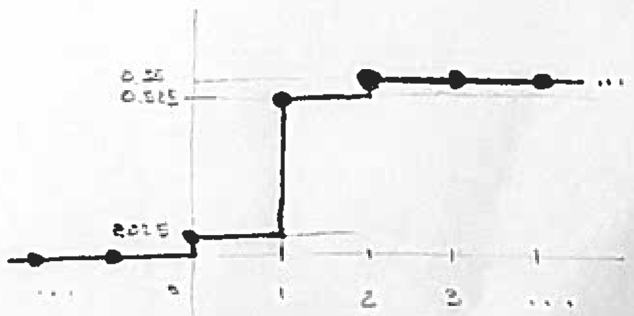
$$y_k = 0.55 \qquad k \geq 0$$

(e.iii) G is a finite impulse response filter therefore it reaches steady state in $M = 2$ steps. For $u_{-2} = u_{-1} = 0$, using the difference equation or the impulse response of the filter, we get

$$\begin{aligned} y_0 &= \frac{0.08}{\pi} \simeq 0.025 \\ y_1 &= \frac{0.08}{\pi} + \frac{1}{2} \simeq 0.525 \\ y_2 &= \frac{0.08}{\pi} + \frac{1}{2} + \frac{0.08}{\pi} \simeq 0.55 \\ y_k &= 0.55 \qquad k > 2 \end{aligned}$$



Sketch of Bode diagram part (d)



Sketch of the complete response part (e)

Question 3

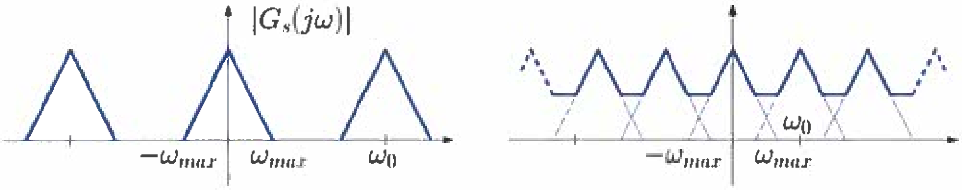
(a) Compute $G_c(s)$ for

$$s = \frac{z-1}{zT} = 10^3\frac{z-1}{z}$$

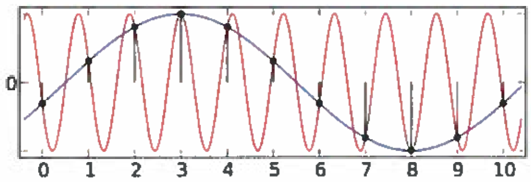
which gives

$$G_d(z) = \frac{10^5}{10^3\frac{z-1}{z}\left(10^6\left(\frac{z-1}{z}\right)^2 + 50\cdot 10^3\frac{z-1}{z} + 10^5\right)} = \frac{10^{-4}z^3}{(z-1)\,(1.15z^2 - 2.05z + 1)}$$

(b) Sampling introduces periodicity in the sampled signal spectrum. If the sampling period does not satisfy Shannon theorem, the sampled signal spectrum overlaps introducing distortion known as aliasing. For instance, let $[0, \omega_{max}]$ be the signal spectrum and ω_0 be the sampling frequency: the left figure below shows a sampled spectrum (periodic) not affected by aliasing; the right one shows aliasing introduced by sampling at insufficient sampling frequency.



In the time domain aliasing occurs when sampling of two different analog signals gives the same sequence of samples, producing the same signal at reconstruction. This case is illustrated in the figure below.



From Shannon sampling theorem the maximum admissible sampling period for the bandwidth 20 − 200Hz is $T_{max} = 1/400\text{s} = 0.0025s$. The sampling period $T < T_{max}$ is thus sufficient to avoid aliasing.

(c) The subwoofer resonance is centered at about 50Hz. At that frequency the filter $F(z)$ makes a sharp correction of $-16dB$, removing the resonance peak.

For instance, the finite impulse response filter $F(z)$ has two zeros $Z = 0.975e^{j0.1\pi}$ and $Z^* = 0.975e^{-j0.1\pi}$ near the unit circle:

- $|Z - e^{j0.1\pi}| = 0.025$ and $|Z^* - e^{j0.1\pi}| \leq 0.2\pi$ (using the arc length as a straightforward bound). It follows that

$$|F(e^{j0.1\pi})| \leq 10\cdot 0.025\cdot 0.2\pi \simeq 0.16 \simeq -16\text{dB} \ ;$$

- The sampling period $T = 0.001$ gives a maximal spectrum bandwidth of 500Hz. Thus, the normalized frequency 0.1π corresponds to the frequency

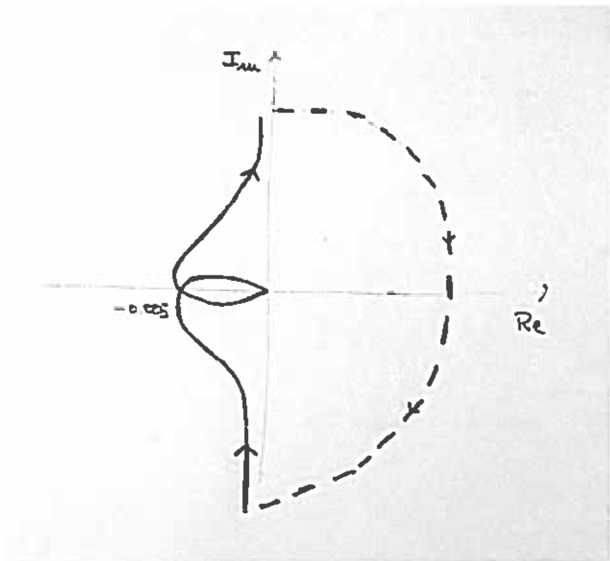
$$\omega = 0.1\pi \cdot \frac{500}{\pi} = 50\text{Hz} .$$

At lower frequencies, the filter magnitude is close to unity, not affecting the subwoofer behavior (for example $F(1) \simeq 10 \cdot 0.1\pi \cdot 0.1\pi \simeq 1$).

(d) Nyquist criterion: the closed-loop will be stable if (and only if) the number of counter clockwise encirclements of the $-1/k$ point by $G_d(e^{j\theta})$ as θ increases from 0 to 2π is equal to the number of open-loop unstable poles.

For the complete Nyquist diagram please refer to the figure below. We have added a large semicircular arc in the clockwise direction connecting the upper branch to the lower branch since $G_d(z)$ has a pole in 1.

The Nyquist diagram crosses the real axis at about -0.005. The open loop transfer function has no unstable poles therefore we look for gains k for which the Nyquist diagram does not encircle $-1/k$. By the Nyquist criterion the closed loop is stable for $0 \leq k < 200$ (no encirclement). The closed loop is unstable for $k > 200$ and for $k < 0$ (one encirclement).



Question 4.

a) (i) Autocorrelation fn. $r_{XX}(t_1,t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right]$

(ii) For $\{X\}$ to have a PSD, the autocorrelation function needs to be a function of the lag $\tau = t_2 - t_1$ only, i.e. $r_{XX}(t_1,t_2) = r_{XX}(\tau)$.

(iii) If X is WSS then $r_{XX}(-\tau) = r_{XX}(\tau)$.

PSD is defined as

$$S_X(\omega) = \int_{-\infty}^{\infty} r_{XX}(\tau)e^{-j\omega\tau}\,d\tau.$$

$$\cancel{= \int_{-\infty}^{\infty} r_{XX}(\tau)\left[\cos(\omega\tau) - j\sin(\omega\tau)\right]dt}$$

$$= \int_0^{\infty} r_{XX}(\tau)e^{-j\omega\tau}\,d\tau + \int_{-\infty}^{0} r_{XX}(\tau)e^{-j\omega\tau}\,d\tau$$

$$= \int_0^{\infty} r_{XX}(\tau)e^{-j\omega\tau}\,d\tau + \int_0^{\infty} r_{XX}(-\tau)e^{j\omega\tau}\,d\tau$$

$$= \int_0^{\infty} r_{XX}(\tau)\left[e^{-j\omega\tau} + e^{j\omega\tau}\right]d\tau$$

$$= 2\int_0^{\infty} r_{XX}(\tau)\cos(\omega\tau)\,d\tau$$, which is real-valued.

(iv) From the calculation in (iii) we have

$$S_X(\omega) = \int_0^{\infty} r_{XX}(\tau) e^{-j\omega\tau} + \int_0^{\infty} r_{XX}(-\tau) e^{j\omega\tau} \, d\tau .$$

suppose $r_{XX}(-\tau) = -r_{XX}(\tau)$.

Then $$S_X(\omega) = \int_0^{\infty} r_{XX}(\tau) \left[e^{-j\omega\tau} - e^{j\omega\tau} \right] d\tau$$

$$= -2j \int_0^{\infty} r_{XX}(\tau) \sin(\omega\tau) \, d\tau$$

which is purely imaginary.

Therefore an antisymmetric autocorrelation function $r_{XX}(-\tau) = -r_{XX}(\tau)$ would give a purely imaginary PSD.

b) (i) The white noise process $\{W(t)\}$ has infinite power and is therefore not physically realisable. Therefore the model must be an approximation.

(ii) Let $v = \dot{x}$ be particle velocity, so

$$m\dot{v} + \gamma v = w(t).$$

Frequency response of v is given by Fourier transform :

$$\tilde{v}(\omega) = \frac{1}{j\omega m + \gamma} \cdot \tilde{w}(\omega) . = H_v(\omega)\, \tilde{w}(\omega).$$

From lectures we know

$$S_v(\omega) = |H_v(\omega)|^2 S_w(\omega).$$

$$= \frac{1}{\omega^2 + (\gamma/m)^2} S_w(\omega)$$

$$= \frac{W_0}{\omega^2 + (\gamma/m)^2}$$

Now $r_{vv}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_v(\omega) e^{j\omega\tau} d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{W_0 e^{j\omega\tau}}{\omega^2 + (\gamma/m)^2} d\omega .$$

(iii) Mean square velocity is given by $r_{vv}(0)$

Applying filter $H(\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & \text{otherwise} \end{cases}$

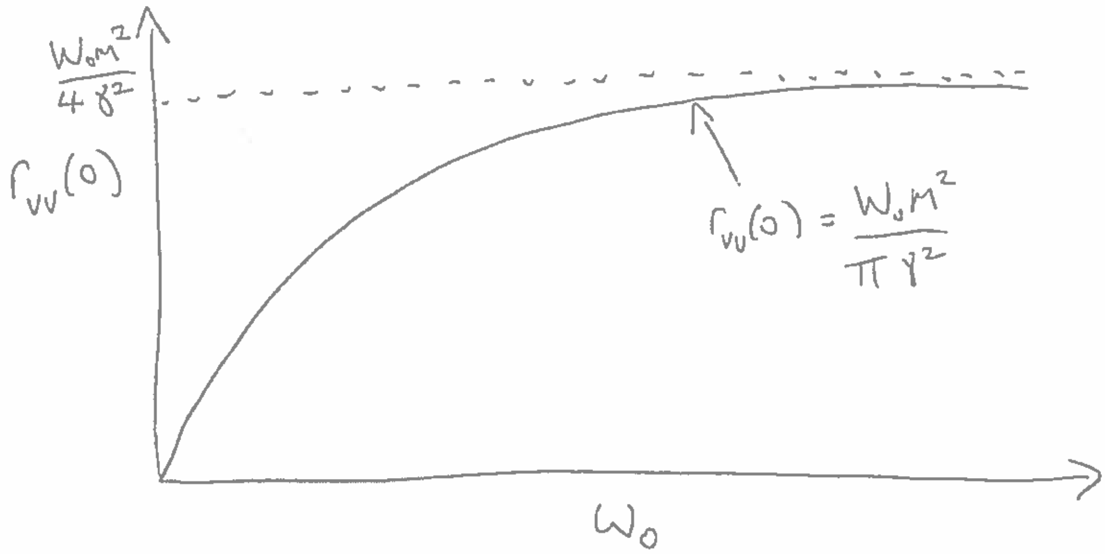
we get : $r_{vv}(0) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{W_0}{\omega^2 + (\gamma/m)^2} d\omega$

$$r_{vv}(0) = \frac{W_0}{2\pi} \int_{-\omega_0}^{\omega_0} \frac{m^2/\gamma^2}{\left(\frac{\omega m}{\gamma}\right)^2 + 1} \, d\omega$$

$$= \frac{W_0 m^2}{2\pi \gamma^2} \left[\tan^{-1}\left(\frac{m\omega}{\gamma}\right) \right]_{-\omega_0}^{\omega_0}$$

$$= \frac{W_0 m^2}{\pi \gamma^2} \tan^{-1}\left(\frac{m\omega_0}{\gamma}\right)$$

(iv) Using expression for $r_{vv}(0)$ in (iii) :



As $\omega_0 \to \infty$, mean square velocity asymptotes to $\frac{W_0 m^2}{4\gamma^2}$. Therefore model is accurate for high cutoff frequencies. Thus high frequency component of the white noise process is not important anyway.