1) a) Observer: $\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x})$ $\ast$

$= (A - LC)\hat{x} + Bu + Ly$

can place poles of $A - HC \Leftrightarrow$ observable

Controller: $\dot{x} = Ax + Bu \quad u = -Fx$

$\Rightarrow \dot{x} = (A - BF)x$

can place poles of $A - BF \Leftrightarrow$ controllable

Together at $u = -F\hat{x}$ + $\ast$

poles $= \lambda(A - BF) \cup \lambda(A - LC)$

ie separation

b) i) $M\ddot{z} = u$

$\underline{x} = \begin{matrix}\dot{z} \\ z\end{matrix} \Rightarrow \dot{\underline{x}} = \begin{bmatrix}\ddot{z} \\ \dot{z}\end{bmatrix} = \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\begin{bmatrix}\dot{z} \\ z\end{bmatrix}$

$+ \begin{bmatrix}1/M \\ 0\end{bmatrix}u$

$z = x_2 = [0 \; 1]\underline{x}$

ii) $\dot{\hat{x}} = \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\hat{x} + \begin{bmatrix}1/M \\ 0\end{bmatrix}u + \begin{bmatrix}h_1 \\ h_2\end{bmatrix}(z - [0 \; 1]\hat{x})$

$= \begin{bmatrix}0 & -h_1 \\ 1 & -h_2\end{bmatrix}\hat{x} + \begin{pmatrix}1/M \\ 0\end{pmatrix}u + \begin{bmatrix}h_1 \\ h_2\end{bmatrix}y$

iii) $(sI - A)^{-1} = \begin{pmatrix}s & h_1 \\ -1 & s+h_2\end{pmatrix}^{-1} = \dfrac{\begin{pmatrix}s+h_2 & -h_1 \\ 1 & s\end{pmatrix}}{s^2 + h_2 s + h_1}$

$\bar{x}_1 = \dfrac{(s+h_2)/M\,\bar{u} + h_1 s\,\bar{z}}{s^2 + h_2 s + h_1}$

if $\bar{z} = \frac{1}{Ms^2} \bar{u}$

the $\bar{X}_1 = \dfrac{\left((s+h_2)/M + \frac{h_1}{Ms}\right) \bar{u}}{s^2 + h_2 s + h_1}$

$= \dfrac{s^2 + h_2 s + h_1}{s^2 + h_2 s + h_1} \cdot \frac{1}{Ms} \bar{u} = \frac{1}{Ms} \bar{u}$

$\Rightarrow$ exact if same initial conditions.

asymptotically exact if $h_1, h_2 > 0$

iv) if $\bar{z} = \cancel{\frac{1}{M_{true} s^2}}$ $\frac{1}{M' s^2}$

$\bar{X}_1 = \dfrac{s^2 + h_2 s}{s^2 + h_2 s + h_1} \cdot \frac{1}{Ms} \bar{u} + \dfrac{h_1}{s^2 + h_2 s + h_1} \cdot \frac{1}{M's} \bar{u}$

For greater accuracy, h_1 small but need h_1 large for ~~g~~ speed of response.

Q1. Examiner's Comment:

This was a question on state-space observer design. There were not many serious attempts at the very last part of the question, where the real system differed from the system that the observer was designed for, but the earlier parts of the question were well answered in general.

2) a) Bookwork

b) i) $u_e = x_{2e}$

$x_{1e} = \pm 1$

ii) let $x_1 = x_{1e} + \delta$.

$\Rightarrow x_1^2 = x_{1e}^2 + 2\delta x_{1e}$

$= 1 + 2\delta x_{1e}$

$\Rightarrow \dot{\delta} = -x_2 + u$

$\dot{x}_2 = 2\delta x_{1e}$

iii) when $u = 0$

$$\frac{\dot{x}_2}{\dot{\delta}} = \frac{2\delta x_{1e}}{-x_2}$$

$u = 0$

$$\begin{pmatrix} \dot{\delta} \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x_{1e} & 0 \end{pmatrix} \begin{pmatrix} \delta \\ x_2 \end{pmatrix}$$

$x_{1e} = 1$ marginally stable

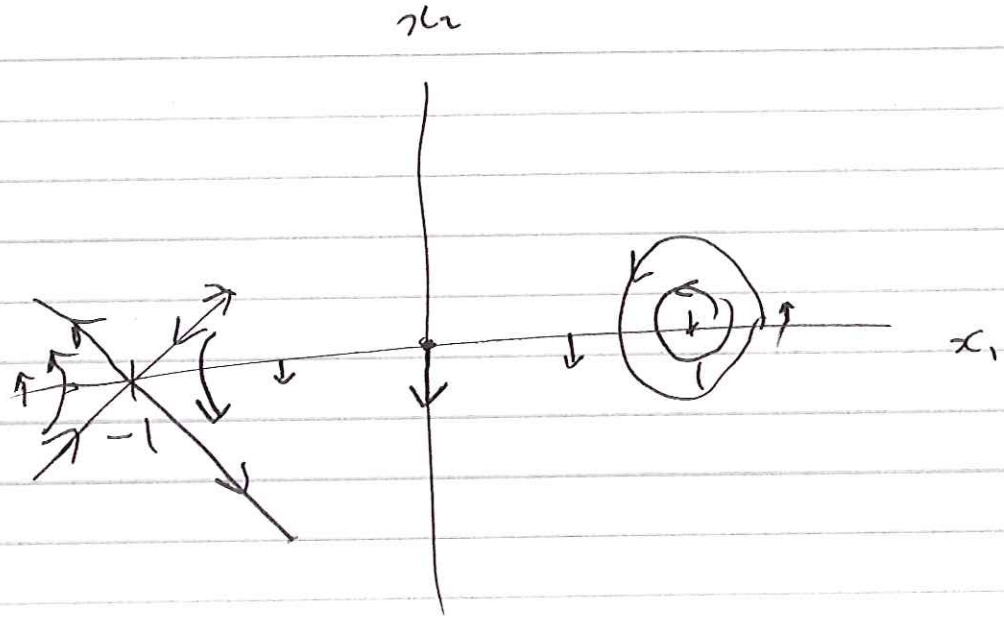
$x_{1e} = -1$ eig $= \frac{1}{\sqrt{2}}, \frac{1}{-\sqrt{2}}$ ← unstable

unstable

$x_{1e} = 1/2$

is saddle

Q(b) 2)
cont



Q2. Examiner's Comment:

This was a question on nonlinear systems, which required sketching of state-plane trajectories as well as a discussion of the uses of linearisation. This was a deliberately straightforward question, as although state-plane trajectories have been an integral part of the lectured material they have not appeared in the exam in recent years or, indeed, in examples papers until this year. The question was very well answered overall.

3) a) $C_1 \dot{x}_1 = \dfrac{u - x_1}{R_1}$

$C_2 \dot{x}_2 = \dfrac{x_1}{R_2}$

$C_3 \dot{x}_3 = \dfrac{x_1 + x_2 - x_3}{R_3}$

b) $(sI - A)^{-1} = \begin{pmatrix} s + 1/\tau_1 & 0 & 0 \\ -1/\tau_2 & s & 0 \\ -1/\tau_3 & -1/\tau_3 & s + \frac{1}{\tau_3} \end{pmatrix}^{-1}$

want $[0 \quad 0 \quad 1] \, (sI - A)^{-1} \begin{pmatrix} 1/\tau_1 \\ 0 \\ 0 \end{pmatrix}$ ie 3, 1 element

ie $\dfrac{s/\tau_3 + \frac{1}{\tau_2 \tau_3}}{(s + \frac{1}{\tau_1}) \, s \, (s + 1/\tau_3)} = \dfrac{\tau_1 (s\tau_2 + 1)}{(s\tau_1 + 1) \, s \, (s\tau_3 + 1)}$

c) $B \;\, AB \;\, A^2B = \begin{bmatrix} \frac{1}{\tau_1} & -1/\tau_1^2 & \frac{1}{\tau_1^3} \\ 0 & \frac{1}{\tau_1 \tau_2} & -\frac{1}{\tau_1^2 \tau_2} \\ 0 & \frac{1}{\tau_1 \tau_3} & -\frac{1}{\tau_1^2 \tau_3} + \underbrace{\frac{1}{\tau_1 \tau_2 \tau_3} - \frac{1}{\tau_1 \tau_3^2}}_{\text{cancel if } \tau_2 = \tau_3} \end{bmatrix}$

2) not controllable if $\tau_2 = \tau_3$
reachable states of form $\begin{pmatrix} a \\ b \\ b \end{pmatrix}$

d) C $\quad 0 \quad 0 \quad 1$

CA $\quad \frac{1}{\tau_3} \quad \frac{1}{\tau_3} \quad -\frac{1}{\tau_3}$

CA^2

$$-\frac{1}{\tau_1\tau_3} + \frac{1}{\tau_2\tau_3} - \frac{1}{\tau_3^2} \quad -\frac{1}{\tau_3^2} \quad \frac{1}{\tau_3^2}$$

loses rank if $\tau_1 = \tau_2$

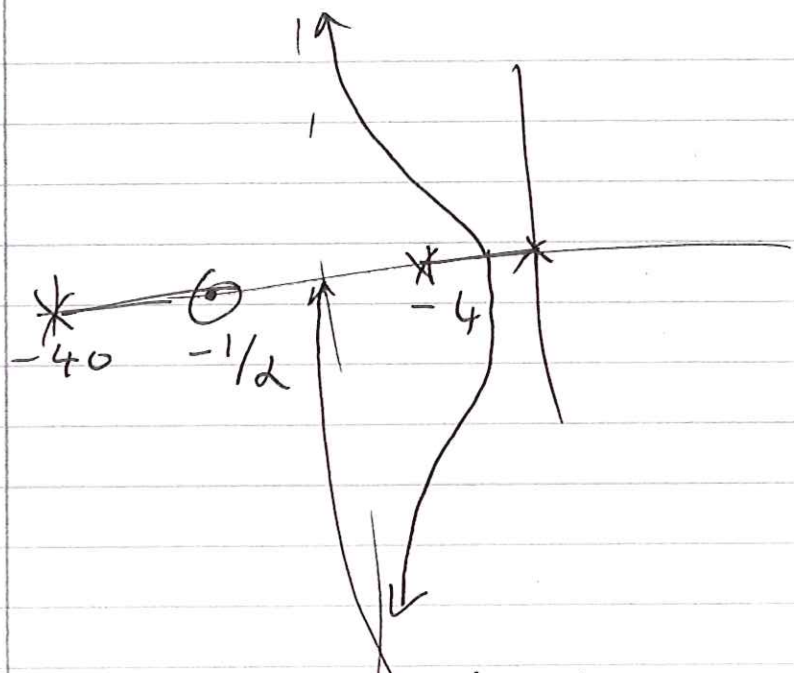
can't see $\begin{pmatrix} -\tau \\ \tau \\ 0 \end{pmatrix}$

e) if $\tau_2 = \tau_3$ or τ_1 then get cancellation.

Q3. Examiner's Comment:

This was a question on observability and controllabilty for an op-amp circuit. It was very well done on the whole, with no particular stumbling blocks. A number of candidates wrote notes complaining about the inclusion of electrical engineering material (for the initial \show" that the system was described by the given transfer function), although it required no more than the ideal op-amp material in Part IA, and similar material does appear in examples papers for this course.

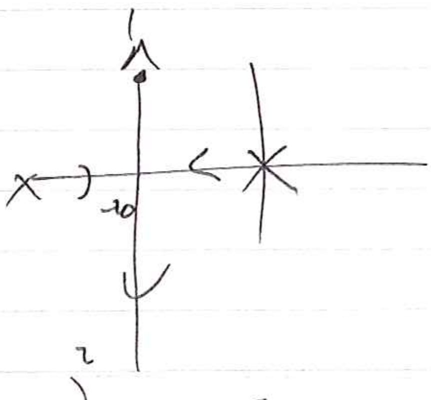
4) a)



$$\frac{-4 - 40 + \frac{1}{\alpha}}{2} = -22 + \frac{1}{2\alpha}$$

b) Closer α is to 0.25, further the asymptote is to the [illegible] legs so the response can be made faster for the same damping ratio

c) $\alpha = 0.25$

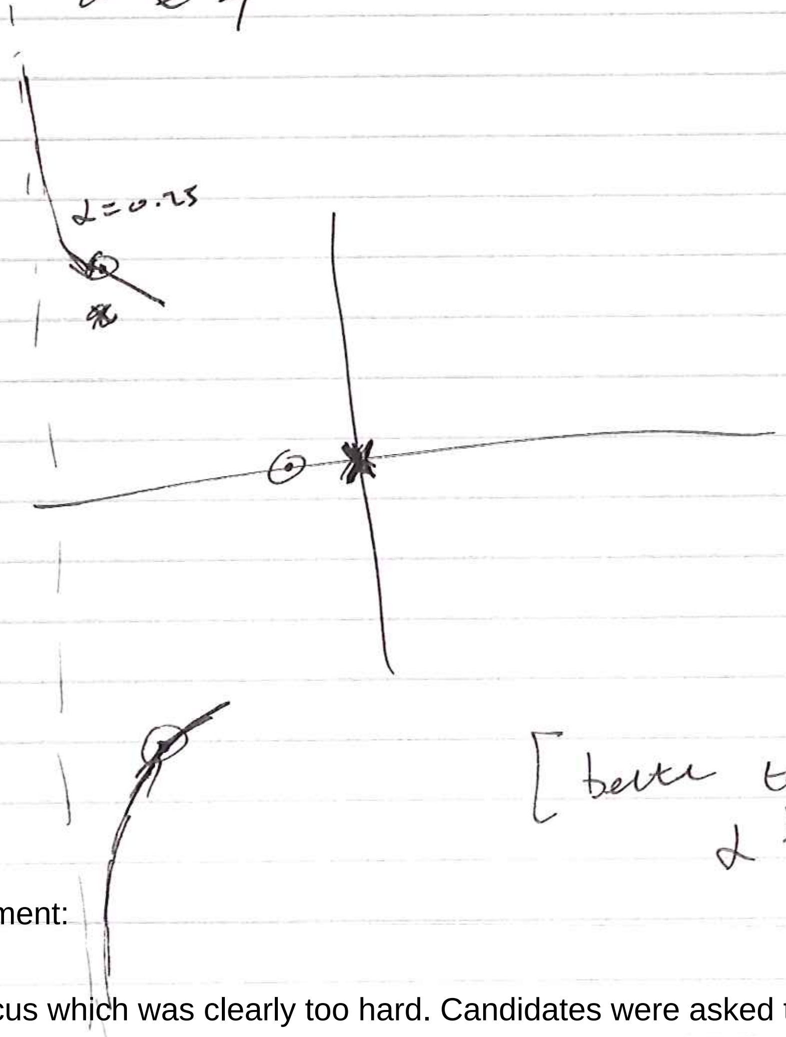


$$K = \frac{(\sqrt{2}\cdot 20)^2}{40} = 20$$

d) $1 + k \dfrac{(\alpha s+1)\,0.5}{s(0.25s+1)(0.025s+1)} = 0$

$$\frac{s(0.25s+1)(0.025s+1)}{0.5k} + \alpha s + 1 = 0$$

$$\frac{1 + \dfrac{s(0.25s+1)(0.025s+1)}{\cancel{0.25s}\ \cancel{\alpha}\ 0.5k}}{\cancel{0.5\,\alpha}} \cdot \frac{1}{\alpha s} + 1 = 0$$



[better to offset α by 0.25]

Q4. Examiner's Comment:

A question on root locus which was clearly too hard. Candidates were asked to sketch representative root locus diagrams over a range of a parameter, and the arguments required for a complete solution were rather subtle. Nevertheless, candidates did have many opportunities to show their understanding of the underlying techniques, which was generally sound.