3F2 2016

1) a) i) $$\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\\ \dot{v}_3\end{bmatrix} = k\begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 0\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$C = [0 \quad 0 \quad 1]$$
$$CA = [0 \quad k \quad -k]$$
$$CA^2 = [k^2 \quad * \quad *]$$

$\Rightarrow \det\begin{bmatrix}C\\CA\\CA^2\end{bmatrix} = k^3 \neq 0$

$\Rightarrow$ observable

ii) $$CA^3 = [k^3 \quad * \quad * \quad *]$$

$$\det\begin{pmatrix}C\\ \vdots\\ CA^3\end{pmatrix} = k^3 . k^2 . k . 1 \neq 0$$

In general $CA^{n-1} = [k^{n-1} \quad * \quad * \quad \dots \quad *]$

$\Rightarrow \det\begin{bmatrix}C\\ \vdots\\ CA^{n-1}\end{bmatrix} = k^{n-1}.k^{n-2}. \dots .k^0 \neq 0$

$\Rightarrow$ observable for any length n

b) $$C = [0 \quad 0 \quad 1]$$

$$A - HC = \begin{bmatrix}0 & 0 & -h_1\\ 2 & -2 & -h_2\\ 0 & 2 & -2-h_3\end{bmatrix}$$

$$\Rightarrow (sI - A + HC) = \begin{bmatrix}s & 0 & h_1\\ -2 & 2+s & h_2\\ 0 & -2 & 2+h_3+s\end{bmatrix}$$

$$\det(sI - A + HC) = s((1+s)(1+h_3+s) + h_2) + h_1$$

$$= s^3 + (2+h_3)s^2 + (1+h_3+h_2)s + h_1$$

$$\equiv (s+1)^3$$

$$\Rightarrow h_3 = -1, \quad h_2 = \tfrac{1}{2}, \quad h_1 = \tfrac{1}{4}$$

Hence, observer has the form

$$\dot{\hat{v}}_1 = u + (v_3 - \hat{v}_3)$$
$$\dot{\hat{v}}_2 = \hat{v}_1 - \hat{v}_2 + (v_3 - \hat{v}_3)$$
$$\dot{\hat{v}}_3 = \hat{v}_2 - \hat{v}_3 + (v_3 - \hat{v}_3)$$

c) If $u = 2(v - v_1)$ then all three poles will be at $s = -2$. If estimated states are used instead then poles will be at the union of the poles with state feedback and the observer poles.

$$\Rightarrow \quad s = -1, -1, -1, -2, -2, -2$$

7) a) i) Bookwork

ii) via Grammian or controllability matrix

b) i) $\dot{x}_1 = u - k_1(x_1 - x_2)$ etc

ii) $[B \ AB \ A^2B] = \begin{bmatrix} 0 & k_1 & -k_1(2k_1+k_2+k_3) \\ 1 & -(k_1+k_2+k_3) & k_1^2+(k_1+k_2+k_3)^2+k_3^2 \\ 0 & k_3 & -k_3(k_1+k_2+2k_3) \end{bmatrix}$

$$\det(\,"\,) = k_1k_3\left((k_1+k_2+2k_3)-(2k_1+k_2+k_3)\right)$$
$$= k_1k_3(k_3-k_1)$$
$$\neq 0 \text{ if } k_1, k_3 > 0 \ \& \ k_1 \neq k_3$$

iii) $(sI-A) = \begin{matrix} s+1 & -1 & 0 \\ -1 & s+2 & -1 \\ 0 & -1 & s+1 \end{matrix}$

$$\det = (s+1)\left((s+1)(s+2)-1\right) + -1.(s+1)$$
$$= (s+1)(s^2+3s) = s(s+1)(s+3)$$

want 2^{nd} column of $(sI-A)^{-1}$

ie $\quad [s+1, (s+1)^2, (s+1)]^T / \det \quad$ (using cofactors)

$$= \begin{bmatrix} \frac{1}{s(s+3)} \\ \frac{s+1}{s(s+3)} \\ \frac{1}{s(s+3)} \end{bmatrix}, \text{ so need step response of } \begin{bmatrix} \frac{1}{s+3} \\ \frac{s+1}{s+3} \\ \frac{1}{s+3} \end{bmatrix}$$

for x_1, x_3 get $\frac{1}{3}(1-e^{-3t})$

for x_2, $\frac{s+1}{s+3} = 1 - \frac{2}{3+s} \rightleftharpoons 1 - \frac{2}{3}(1-e^{-3t})$

$$= \frac{1}{3} + \frac{2}{3}e^{-3t}$$

$$\{B \quad AB \quad A^2B\} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -2 & 6 \\ 0 & 1 & -3 \end{bmatrix}$$

$\Rightarrow$ Reachable states are of form $\begin{bmatrix} b \\ a-2b \\ b \end{bmatrix}$

$\Rightarrow x_1 = x_3$ for all t

3) a) $\dot{x}_1 = x_2$

$\dot{x}_2 = -2x_2 - 8\sin x_1$

$x_2 = 0, \quad \sin x_1 = 0 \Rightarrow x_1 = n\pi$

b) $\dfrac{\partial f_2}{\partial x_1} = -8\cos x_1 = \begin{cases} -8, & x_1 = n\pi, n \text{ even} \\ 8, & x_1 = n\pi, n \text{ odd} \end{cases}$

<u>n even</u>

$$A = \begin{bmatrix} 0 & 1 \\ -8 & -2 \end{bmatrix}$$

$$\det \begin{bmatrix} \lambda & -1 \\ 8 & \lambda+2 \end{bmatrix} = \lambda^2 + 2\lambda + 8 = 0$$

$$\Rightarrow \lambda = -1 \pm \sqrt{7}j$$

$\Rightarrow$ <u>stable</u>

<u>n odd</u>

~~det =~~

$$A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \lambda^2 + 2\lambda - 8 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4+32}}{2} = -1 \pm \sqrt{9}$$

$$= -4, 2$$

$\Rightarrow$ unstable

<u>eigenvectors</u>

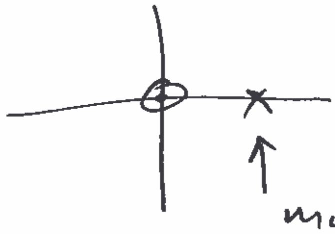
$$\begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_2 = -4x_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_2 = 2x_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

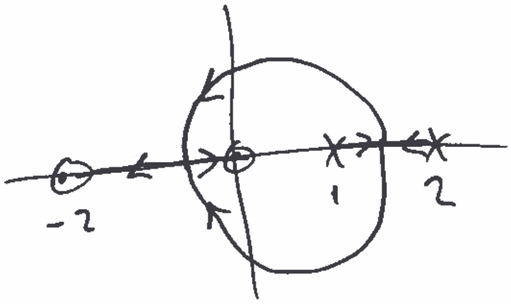
4) a) $(s-2)\bar{x} = s\bar{u} \Rightarrow \bar{x} = \frac{s}{s-2}\bar{u}$



must move this pole off real axis to get into LHP. This requires an unstable controller (ie another pole in RHP)

Implications: when and how do you switch it on.

b)



CLCE $1 + \frac{ks(s+2)}{(s-1)(s-2)}$ ie $s^2 - 3s + 2 + ks^2 + 2ks$

$\Rightarrow$ stable for $2k > 3$, $k > \cancel{\frac{2}{3}}$ $\frac{3}{2}$

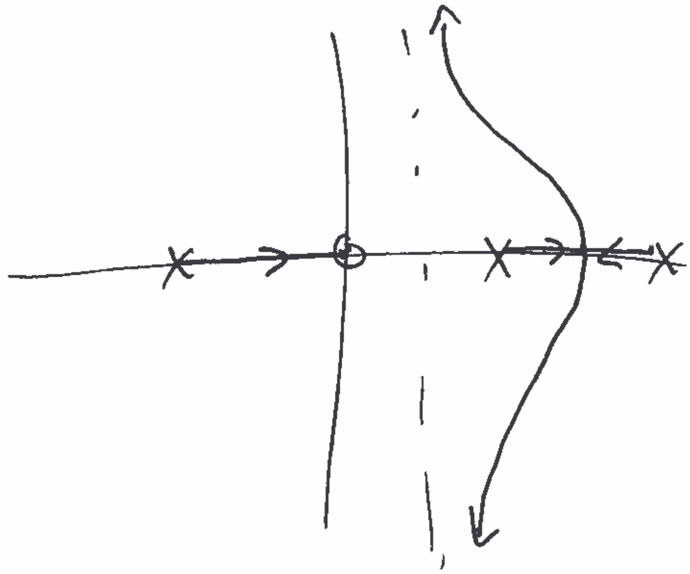
$$\frac{1+k}{2}s^2 + \frac{2k-3}{2}s + 1$$

need $\left(\frac{2k-3}{2}\right)^2 = 4 \cdot \frac{1+k}{2} \Rightarrow (2k-3)^2 = 8 + 8k$

$\Rightarrow 4k^2 - \cancel{14}\,20\,k + 1 = 0 \Rightarrow k = \cancel{3.43}$ 4.95

c)

i)



asymptotes at $\dfrac{-1 \quad +1 \quad +2}{\cancel{4}\ 2} = 1/2$

ii) need $\dfrac{-a+1+2}{\cancel{4}\ 2} < 0 \Rightarrow a > 3$

(ie asymptote in LHP)