1 a) Bookwork

b) i) Let T_1, T_2, T_3 be 3 outlet temps, and T_0 1st inlet temperature

$$\dot{T}_1 = (k_a - k_b) T_0 - (k_a + k_b) T_1$$
$$\dot{T}_2 = (k_a - k_b) T_1 - (k_a + k_b) T_2$$
$$\dot{T}_3 = (k_a - k_b) T_2 - (k_a + k_b) T_3$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} -(k_a+k_b) & 0 & 0 \\ k_a-k_b & -(k_a+k_b) & 0 \\ 0 & k_a-k_b & -(k_a+k_b) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} k_a-k_b \\ 0 \\ 0 \end{bmatrix} T_0$$

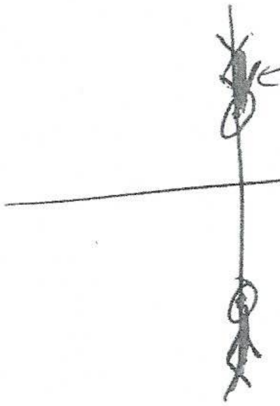
ii) $$T_1 = \overbrace{\left(\frac{k_a - k_b}{k_a + k_b}\right)}^{\alpha} T_0 \quad , T_2 = \alpha T_1 = \alpha^2 T_0$$
$$\Rightarrow T_3 = \alpha^3 T_0$$

iii) $$P = \begin{bmatrix} k_a-k_b & -(k_a+k_b)(k_a-k_b) & ? \\ 0 & (k_a-k_b)^2 & ? \\ 0 & 0 & (k_a-k_b)^3 \end{bmatrix}$$ not needed

$\Rightarrow \det(P) = (k_a - k_b)^6 \Rightarrow$ controllable if $k_a \neq k_b$ ($k_a = k_b$ is not interesting, as input has no effect)

iv) Controllable means can steer state anywhere in any finite time, but not necessarily hold it there. In this case can steer to any equilibrium state arbitrarily quickly & then hold it there with T_0 satisfying eqns of (ii)

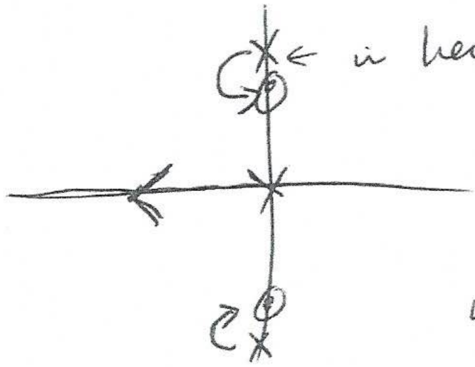
2) a)



← in here $\angle = 90 - (-90)$
$= 180 \Rightarrow$
on RL

$\Rightarrow$ marginally stable for all K

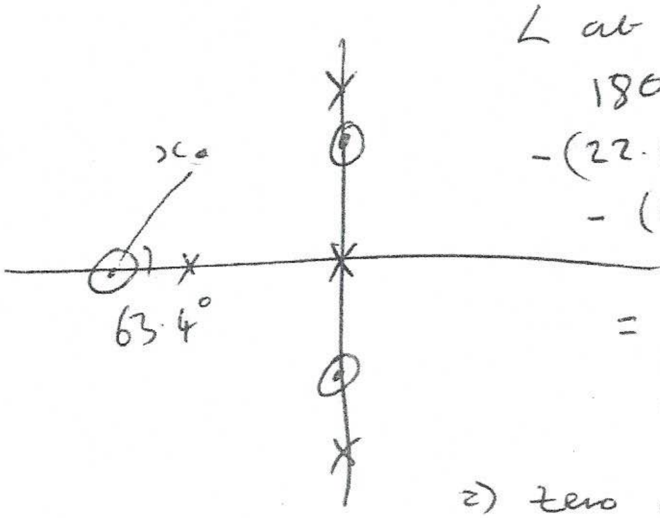
b)



← in here $\angle = 90 - (-90) - 90$
$= 90$
Hence need to move to the left to increase contribution from zero

$\Rightarrow$ stable for all k

c)



$\angle$ at x =
$180 + 116.57$
$- (22.5 + \cancel{180}) - 135$
$- (90 + 22.5) - 90$
+ contrib from zero
$= -153.4 - 90 + \theta = -180$

$\Rightarrow \theta = 63.4^\circ$

$\Rightarrow$ zero at $-1 - \dfrac{1}{\tan\theta}$

$= -1.5$

$$K(s) = k\,\frac{(s + 1.5)}{(s + 1)}$$

3 a) $m\ddot{x} = F - mg$

$$\ddot{x} = \frac{F}{m} - g$$

$x_1 = x$

$x_2 = \dot{x}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \underbrace{\frac{K}{m}\left(\frac{1}{x_1^2} - \frac{1}{(x_1+L)^2}\right) i(t)}_{f(x_1, i)} - g \end{cases}$$

b) $$i_{eq} = \frac{g}{\frac{K}{m}\left(\frac{1}{X^2} - \frac{1}{(X+L)^2}\right)}$$

$$\frac{\partial f}{\partial i} = \frac{K}{m}\left(\frac{1}{X^2} - \frac{1}{(X+L)^2}\right)$$

$$\frac{\partial f}{\partial x_1} = \frac{K\, i_{eq}}{m}\left(\frac{-2}{X^3} - \frac{-2}{(X+L)^3}\right)$$

$$= \frac{2K\, i_{eq}}{m}\left(\frac{1}{(X+L)^3} - \frac{1}{X^3}\right) = \frac{-2K\, i_{eq}}{m}\,\frac{(X+L)^3 - X^3}{X(X+L)}$$

$$= \frac{d}{dt}\begin{Bmatrix} \delta x_1 \\ \delta x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \partial f/\partial x_1 & 0 \end{bmatrix}\begin{Bmatrix} \delta x_1 \\ \delta x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{K}{m}\left(\frac{1}{X^2} - \frac{1}{(X+L)^2}\right) \end{bmatrix} i$$

c) $\frac{\partial f}{\partial x_1} = -433.3$, $\frac{\partial f}{\partial i} = 355.56$, $\lambda(A - BK) = -1 \pm j$

ie $i = 0.0281 + 1.213x - 0.0056\, dx/dt$ $\Rightarrow K = [-1.213 \;\; 0.0056]$

d) $X = 0.06 \Rightarrow i_{eq} = 0.0419$, $\partial f/\partial x_1 = -367.42$, $\frac{\partial f}{\partial i} = 238.72$

Same $K \Rightarrow \lambda = -0.67 \pm 8.8j$

c) (more detail)

$$\det(\lambda I - A + BK) = \begin{vmatrix} \lambda & -1 \\ -\frac{\partial f}{\partial x_1} + k_1 \frac{\partial f}{\partial i} & \lambda + k_2 \frac{\partial f}{\partial i} \end{vmatrix}$$

$$= \lambda^2 + k_2 \frac{\partial f}{\partial i} \lambda - \frac{\partial f}{\partial x_1} + k_1 \frac{\partial f}{\partial i} \qquad (*)$$

compare coeffs to $\lambda^2 + 2\lambda + 2$ to get k_1, k_2

d) find roots of $(*)$ for new $\frac{\partial f}{\partial x_1}$ & $\frac{\partial f}{\partial i}$

4) a) & b) Bookwork

c) $$\frac{d}{dt}\begin{pmatrix} x \\ x-\hat{x} \end{pmatrix} = \begin{pmatrix} A-BF & BF \\ 0 & A-HC \end{pmatrix}\begin{bmatrix} x \\ x-\hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} d$$

$$y = [C \quad 0]\begin{bmatrix} x \\ x-\hat{x} \end{bmatrix}$$

want $$\begin{pmatrix} sI - A + BF & -BF \\ 0 & sI-A+HC \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} (sI - A + BF)^{-1} & (sI-A+BF)^{-1}BF(sI-A+HC)^{-1} \\ 0 & (sI-A+HC)^{-1} \end{pmatrix}$$

[Give matrix inversion Lemma as hint!)

$$\Rightarrow y(s) = C(sI-A+BF)^{-1}B\left(I - F(sI-A+HC)^{-1}B\right)\bar{d}(s)$$

c.f $C(sI-A+BF)^{-1}B$ with state f/b

d) eigenvalues of $(A-BF)$ and $(A-HC)$

- bookwork