Version GV/3

1 (a) What does it mean for a system to be *controllable*? [10%]

for any $x_0, x_1, t_0, t_1 > t_o$ $\exists$ a $u(t)$ that takes state from x_0 at t_0 to x_1 at t_1.

(b) Describe two different ways in which the controllability of a system can be tested, being careful to define the matrix appropriate to each test. What are the benefits of each test? [20%]

rank of controllability matrix $[A^n B \ldots ABB] = n$ OR rank of controllability grammian $\int_0^\infty e^{A\tau} BB^T e^{A^T\tau}\, d\tau = n$. First test is easier, second matrix is close to singular if it takes large amount of energy to control system in some directions.

(c) A model of the longitudinal dynamics of an aircraft is given by $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ where the elements of the state vector $\underline{x}$ are forward velocity, vertical velocity, pitch rate and pitch angle respectively. The first input is aileron angle and the second is thrust. The state-space matrices are given by

$$A = \begin{bmatrix} 0 & 1.0 & 0 & -3.0 \\ 0 & -3.0 & 0.5 & 0 \\ 1.0 & -10.0 & -4.0 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Find a state feedback gain matrix to place the closed-loop poles at $s = -3.5 \pm j$, $-0.05 \pm 0.05j$, using the thrust input alone. [40%]

$\det(sI - A + B_2[k_1\, k_2\, k_3\, k_4]) = s^4 + (k1 + 7) * s^3 + (7 * k1 + k3 + 17) * s^2 + (17 * k1 + k2/2 + 3 * k3 + k4 + 2.5) * s + 3 * k4 + 9 \implies$ $k_1 = 0.1$, $k_2 = 22.7458$, $k_3 = -3.745$, $k_4 = -2.9779$

(d) Can you infer from your working in part (c) that the system is controllable from the thrust input alone *without* carrying out the tests in part (b)? [10%]

yes, as it always possible to compare coeffs in this way for any pole positions and sequentially solve for the gains (i.e. coeff matrix of linear eqns is triangular). Since controllable $\iff$ can place poles then system must be controllable in each case.

(e) What would be the limitations of controlling the aircraft in this way, and how might the design be improved? [20%]

controllable doesn't mean that you can maintain the system at a desired non zero steady state. In order to accelerate the aircraft to a different forward velocity, for example, would require the use of both inputs

2 Consider the system with transfer function

$$G(s) = \frac{s^2 - 12s + 48}{s^2 + 12s + 48}$$

which is to be controlled in a negative feedback configuration by a controller $K(s)$.

(a) (i) Show that the root locus diagram for this feedback system, for $K(s) = k$, consists of arcs of a circle centred at the origin and consequently [20%]

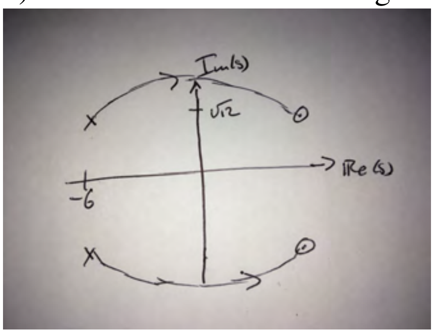
$1 + kG(s) = 0$ for some $k \iff \Im(G(j\omega)) = 0$. Put $s = a + bj$ then

$$G(s) = \frac{a^2 - b^2 + 2abj - 12a - 12bj + 48}{a^2 - b^2 + 2abj + 12a + 12bj + 48}$$

$$= (a^2 - b^2 + 2abj - 12a - 12bj + 48)(a^2 - b^2 - 2abj + 12a - 12bj + 48)/\bullet$$

where the denominator is real. The imaginary part of this is proportional to $b(a^2 + b^2 - 48)$ so either s is real or $|s|^2 = 48$ as required. The RL diagram will rule out real s for $k > 0$.

(ii) sketch the root locus diagram for this system. [15%]



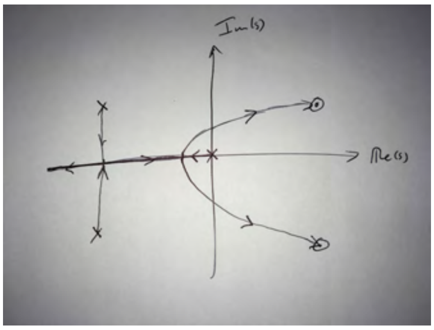
(iii) Describe the behaviour of the feedback system as k is increased from zero to a large positive value. [15%]

slightly oscillatory for $k = 0$, marginally stable for $k = 1$ (oscillations at $\omega = \sqrt{48}$). Unstable (divergent oscillations) for $k > 1$.

(b) Now consider the controller $K(s) = \frac{k}{s}$.

(i) Sketch the root-locus diagram of the new feedback system. [20%]

$-ve$ real axis is on RL (odd number to the right). Potential breakaway points at $\frac{d}{ds}(G(s)/s) = 0$ ie $(s^4 - 24 * s^3 - 48 * s^2 + 1152 * s + 2304)/\ldots = 0$, which is the given polynomial. Only -ve real roots give breakaway points, so RL must have the form shown here, with breakaway points at -2 and -5.9.



(ii) For what range of k are all the poles of the closed-loop system real valued? [20%]
Need k between the values for the two breakawaypoints, ie $k = 1/|G(s)|$ for $s = -2$ and -5.9, ie $0.46 < k < 0.74$.

(c) For this system, which controller is preferable, and why? Are your conclusions surprising? **The integral action controller is preferable, as it can provide greater damping as well as zero steady state error. This is surprising as pure integral action typically *decreases* damping.** [10%]

[You may use the fact that the roots of polynomial $z^4 - 24 * z^3 - 48 * z^2 + 1152 * z + 2304$ are at $z \approx 23.8,\ 8.1,\ -5.9,\ -2.0$]

3 (a) Discuss the role of linearised models in control system design. When is the use of such models justified? [15%]

bookwork: used when the systems is smooth and small perturbations about equilibrium are required. Linear control can ensure perturbations remain small and so assumptions are valid.

(b) Consider the system

$$\ddot{x} + \dot{x} - (1 - x^2)x = u$$

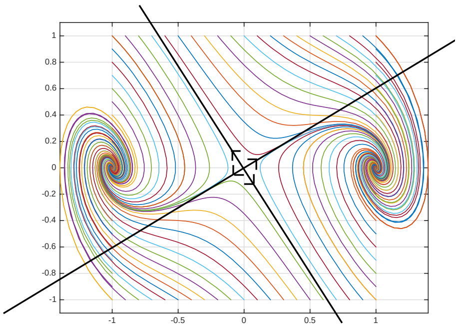
(i) Find all equilibria of this system when $u = 0$. [15%]

$\mathbf{x = -1, 0, 1}$

(ii) Linearise the system about each equilibrium. [20%]

$x_1 = x$, $x_2 = \dot{x}$, so $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_1 - x_1^3 - x_2$ so $df_2/x_1 = 1 - 3x_1^2 = 1$ or -2, so $A_0 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ with e.vector $[1, -1.618]$ corresponding to stable e.value (-1.618) and $[1.618, 1]$ to the unstable one (0.618) and $A_1 = A_{-1} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ with complex stable eigenvalues

(iii) Sketch the state trajectories of this system, being careful to indicate any asymptotes. [20%]



asymptotes shown with slopes 1/1.618 and -1.618/1 for the unstable and stable eigenvalues respectively

(iv) Calculate the u that shifts one of the equilibria to $x = 1.5$, what happens to the other equilibria? [10%]

$u = -x + x^3 = 1.875$. The other equilibria will disappear as $x^3 - x - 1.875 = (x - 1.5)(x^2 + 1.5x + 1.25)$ has only one real root as $1.5^2 < 4 \times 1.25$.

(v) What state will the system be left in if u starts at 0, slowly increases to the value calculated in (iv) and then slowly decreases again to 0? [10%]

one stable equilibrium for $u = 1.875$, so system will go there and will follow the

equilibrium on the positive real axis as u decreases.

(c) Is the use of a model linearised around the origin justified in terms of understanding the overall behaviour of the system described in (b)? [10%]

No! The key characteristic of the system is its bistability, which is not apparent from the linearisation.

4 (a) Explain, with the aid of block diagrams, how the combination of a state observer and estimated state feedback can be used as the basis of control system design, when it is required that the output of the system follows a given reference signal. [30%]

bookwork (lengthy)

(b) A system is given as

$$\dot{x} = 2x + u, \; y = x + w$$

where w represents measurement noise. The input $u = -k\hat{x} + r$, where r is a reference signal and $\hat{x}$ is the estimated state given by an observer with observer gain h.

(i) Find the closed-loop transfer functions from $\bar{r}(s)$ and $\bar{w}(s)$ to $\bar{y}(s)$ in terms of k and h. [30%]

$\boldsymbol{u = -k\hat{x} + r \implies \dot{x} = 2x - k\hat{x} + r.}$

Also $\boldsymbol{\dot{\hat{x}} = 2\hat{x} + u + h(y - \hat{x}) = 2\hat{x} - k\hat{x} + r + h(x + w - \hat{x}).}$

So

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} 2 & -k \\ h & 2-k-h \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r + \begin{bmatrix} 0 \\ h \end{bmatrix} w$$

Using $\boldsymbol{C(sI - A)^{-1}B_1}$**, for** $\boldsymbol{C = [1,0]}$**, gives** $\boldsymbol{T_{r\to y} = 1/(s + k - 2)}$ **and** $\boldsymbol{C(sI - A)^{-1}B_2 + D}$ **for** $\boldsymbol{D = 1}$ **gives** $\boldsymbol{T_{w\to y} = \frac{(s-2)(s+h+k-2)}{(s+h-2)(s+k-2)}}$

(ii) Find the *open-loop* transfer function of the controller, from $\bar{y}(s)$ to $\bar{u}(s)$. [20%]

setting $\boldsymbol{r = 0}$ **we get** $\boldsymbol{\dot{\hat{x}} = 2\hat{x} - k\hat{x} + h(y - \hat{x})}$ **and so** $\boldsymbol{T_{y\to x} = \frac{h}{s-2+k+h}}$ **and hence** $\boldsymbol{T_{y\to u} = \frac{-kh}{s-2+k+h}}$

(iii) Discuss the role of k and h with reference to your answers to (i) and (ii). [20%]

$\boldsymbol{k}$ **and** $\boldsymbol{h}$ **both need to be greater than 2 for stability of the state feedback and the observer. The closed loop poles are then the union of the state f/b and observer poles, i.e.** $\boldsymbol{k-2}$ **and** $\boldsymbol{h-2}$**. Note that the reference response is independent of** $\boldsymbol{h}$**, so increase** $\boldsymbol{k}$ **to make this faster. As long as** $\boldsymbol{h > 2}$ **it only affects the response to the noise** $\boldsymbol{w}$**.**

END OF PAPER