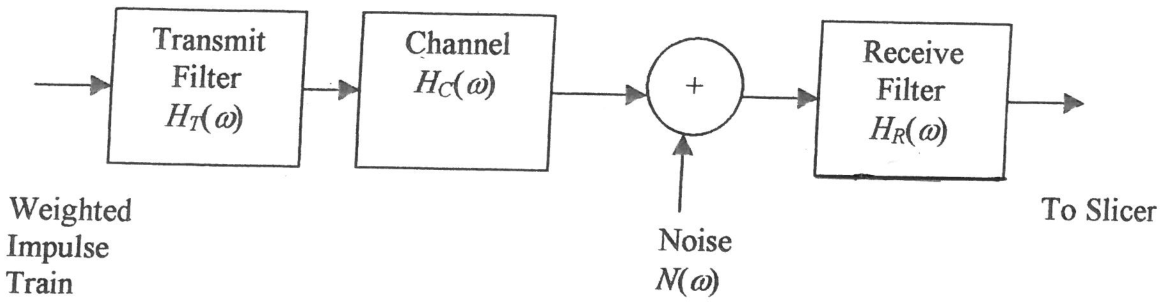
3F4 Data Transmission - 2015/16 Solutions

1 (a) System block diagram:



Usual assumptions about noise:

Gaussian PDF with zero mean.
Uniform PSD (i.e. white). [5]

(b) From Parseval's theorem,

$$E_T = \int_{-\infty}^{\infty} h_T(t)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_T(\omega)|^2 d\omega$$

Also we are told that,

$$H_T(\omega) H_C(\omega) H_R(\omega) = kP_R(\omega)$$

So, substituting for $H_T(\omega)$ in the expression for E_T gives,

$$E_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k^2 |P_R(\omega)|^2}{|H_C(\omega)|^2 |H_R(\omega)|^2} d\omega$$

For a binary PAM system we must maximise,

$$\frac{V_1 - V_0}{2\sigma_v} = \frac{(A_1 - A_0) k p_R(0)}{2\sigma_v}$$

Since A_0, A_1 and $p_R(0)$ are fixed, we must maximise k/σ_v, or alternatively minimise $(\sigma_v/k)^2$, where $(\sigma_v)^2$ is the noise power at the slicer input,

$$\sigma_v^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) |H_R(\omega)|^2 d\omega$$

We now need an expression for k^2. We rearrange the expression for E_T to give,

$$k^2 = \frac{E_T}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|P_R(\omega)|^2}{|H_C(\omega)|^2|H_R(\omega)|^2}d\omega}$$

Substituting for k^2 and $(\sigma_v)^2$ into the expression we wish to minimise gives,

$$\frac{\sigma_v^2}{k^2} = \frac{1}{(2\pi)^2 E_T}\int_{-\infty}^{\infty} N(\omega)|H_R(\omega)|^2 d\omega \int_{-\infty}^{\infty}\frac{|P_R(\omega)|^2}{|H_C(\omega)|^2|H_R(\omega)|^2}d\omega$$

Schwarz inequality states:

$$\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega \int_{-\infty}^{\infty}|G(\omega)|^2 d\omega \geq \left|\int_{-\infty}^{\infty}F(\omega)G(\omega)d\omega\right|^2$$

With equality when $F(\omega) = \lambda G^*(\omega)$ where λ is an arbitrary constant.

So let,

$$F(\omega) = \sqrt{N(\omega)}|H_R(\omega)| \quad \text{and}$$

$$G(\omega) = \frac{|P_R(\omega)|}{|H_C(\omega)||H_R(\omega)|}$$

Giving,

$$\frac{\sigma_v^2}{k^2} = \frac{1}{(2\pi)^2 E_T}\int_{-\infty}^{\infty} N(\omega)|H_R(\omega)|^2 d\omega \int_{-\infty}^{\infty}\frac{|P_R(\omega)|^2}{|H_C(\omega)|^2|H_R(\omega)|^2}d\omega$$

$$\geq \frac{1}{(2\pi)^2 E_T}\left|\int_{-\infty}^{\infty}\sqrt{N(\omega)}\frac{|P_R(\omega)|}{|H_C(\omega)|}d\omega\right|^2$$

All of the terms in the final integral are fixed, therefore the integral has a constant value. Thus $(\sigma_v/k)^2$ is minimised when it is equal to this constant value.

Therefore from the equality condition (which minimises the expression for $(\sigma_v/k)^2$ and with λ=1) we can write,

$$|H_R(\omega)| = \left|\frac{P_R(\omega)}{\sqrt{N(\omega)}H_C(\omega)}\right|^{1/2}$$

[10]

(c) Returning to the Schwarz inequality, we can see,

$$\frac{\sigma_v^2}{k^2} \geq \frac{1}{(2\pi)^2 E_T}\left|\int_{-\infty}^{\infty} \sqrt{N(\omega)}\,\frac{|P_R(\omega)|}{|H_C(\omega)|}\,d\omega\right|^2$$

The expression for $|H_R(\omega)|$ can be rearranged to give (and also substituting $N(\omega)=N_o$),

$$\frac{|P_R(\omega)|}{|H_C(\omega)|} = \sqrt{N_o}\,|H_R(\omega)|^2$$

Consequently, substituting into the expression for $(\sigma_v/k)^2$ gives (with equality condition),

$$\frac{\sigma_v^2}{k^2} = \frac{N_o^2}{(2\pi)^2 E_T}\left|\int_{-\infty}^{\infty} |H_R(\omega)|^2\,d\omega\right|^2$$

We now substitute for $|H_R(\omega)|$ from the question,

$$\frac{\sigma_v^2}{k^2} = \frac{N_o^2}{(2\pi)^2 E_T}\left|\int_{-2\pi/T}^{2\pi/T} \frac{T}{\sqrt{N_o}}\cos^2\left(\frac{\omega T}{4}\right)\,d\omega\right|^2$$

$$\frac{\sigma_v^2}{k^2} = \frac{N_o}{E_T}$$

Now,

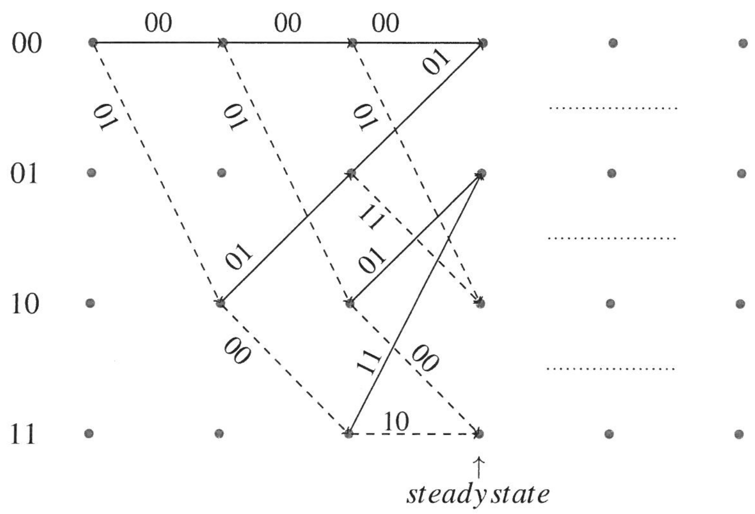
$$P_e = Q\left(\frac{V_1 - V_0}{2\sigma_v}\right) = Q\left(\frac{(A_1 - A_0)kp_R(0)}{2\sigma_v}\right)$$

We know that $A_1=1$, $A_0= -1$ and that $p_R(0)=1$, consequently,

$$P_e = Q\left(\sqrt{\frac{E_T}{N_o}}\right)$$

[5]

2 (a) i) The trellis diagram of the code is shown below. Transitions due to input bit 0 are shown in solid lines, and those due to input 1 are shown in dashed lines. The edges are labeled with the code bits (c_1, c_2) corresponding to the transitions.



ii) The path corresponding to the input 101 on the trellis is 00-10-01-10 and the corresponding codeword is is 010111.

(b) i) Input blocklength= 3, Output blocklength= 5, Rate= $\frac{3}{5}$

ii) The length-5 codeword $\underline{c}$ is generated from the length-3 data vector $\underline{x}$ as $\underline{c} = \underline{x}G$, where G is the 3×5 generator matrix. Thus the codeword corresponding to the input bits 100 is the first row of G. From the mapping $100 \rightarrow 10010$, we conclude that the first row of $\underline{g_1}$ equals 10010. Similarly from the mapping $010 \rightarrow 01011$, we conclude that the second row $\underline{g_2}$ equals 01011.

The codeword corresponding to 001 is the third row $\underline{g_3}$. To find $\underline{g_3}$, observe that by *linearity* of the code, the codeword corresponding to 101 is the sum of the codeword corresponding to 100 and the codeword corresponding to 001. That is,

$$[0\,0\,1\,0\,1] = \underline{g_1} \oplus \underline{g_3} = [1\,0\,0\,1\,0] \oplus \underline{g_3}.$$

Hence the third row $\underline{g_3} = [1\,0\,1\,1\,1] \oplus [1\,0\,0\,1\,0] = [0\,0\,1\,0\,1]$.

iii) Note that G is systematic. When $G = [I_k \mid P]$, the parity check matrix $H = [P^T \mid I_{n-k}]$. Hence the parity check matrix is

$$H = \begin{bmatrix} 1\,1\,0\,1\,0 \\ 0\,1\,1\,0\,1 \end{bmatrix}.$$

iv) The minimum distance d_{min} equals the smallest number of columns of H that add to zero. Since columns 1 and 4 are indentical (as are columns 3 and 5), $d_{min} = 2$, and guaranteed error correcting capability $= \lfloor \frac{d_{min}-1}{2} \rfloor = 0$.

v) A possible syndrome table is

Syndrome	Error pattern
00	00000
01	00100
10	10000
11	01000

(There are other possibilities for the table, but all of them have three correctable weight-one error patterns.) Note can't correct all single bit error patterns.

vi) A correct decision is made only when the error pattern is one of those listed in the syndrome table. Thus the error probability is

$$P_e = 1 - (1-p)^5 - 3p(1-p)^4.$$

3 (a) The complex phasor $p(t)$ allows both amplitude and phase variations of $s(t)$ to be represented in a single time-varying function. Hence if $p(t) = a(t)e^{j\phi(t)}$ then:

$$s(t) = \Re[a(t)e^{j(\omega_C t + \phi(t))}] = a(t)\cos(\omega_C t + \phi(t))$$

so the phase of $p(t)$ controls the phase of $s(t)$ and its ampltitude $a(t)$ controls the amplitude of $s(t)$.

(b) We may write:

$$s(t) = 1/2[p(t)e^{j\omega_C t} + p^*(t)e^{-j\omega_C t}]$$

Now if $p(t)$ transforms to $P(\omega)$ then $p^*(t)$ tranforms to $P^*(-\omega)$.
And so, by frequency shift: $p(t)e^{j\omega_C t}$ transforms to $P(\omega - \omega_C)$ and $p^*(t)e^{-j\omega_C t}$ transforms to $P^*(-(\omega + \omega_C))$.
Hence $s(t)$ transforms to:

$$S(\omega) = \frac{1}{2}[P(\omega - \omega_C) + P^*(-(\omega + \omega_C))]$$

(c) A stream of data impulses with period T_s and random phases may be defined by

$$b(t) = \sum_k b_k \delta(t - kT_s)$$

where $|b_k| = 1$ for all k.
The power spectral density (PSD) of $b(t)$ is then given by:

$$\lim_{T\to\infty} E[|B_T(\omega)|^2]/T = 1/T_s \sum_L R_{bb}(L)e^{(jL\omega T_s)}$$

where $R_{bb}(L)$ is the discrete autocorrelation function of the symbols b_k.
When the symbols b_k are zero mean, with phases that are random and uncorrelated, then: $R_{bb}(L) = E[b_k b_{k-L}] = 1$ if $L = 0$, and $= 0$ if L is nonzero.
Therefore the PSD of $b(t)$ is $\frac{1}{T_s}R_{bb}(0)\exp(0) = 1/T_s$
The given stream of rectangular pulses is equivalent to passing the random impulse stream through a filter, whose impulse response is $g(t) = a_0$ if $0 \le t < T_s$ and whose frequency response is $G(\omega) = a_0 T_s \mathrm{sinc}(\omega T_s/2)e^{-j\omega T_s/2}$
Hence the PSD of the stream of filtered pulses $p(t)$ will be

$$\frac{1}{T_s}|G(\omega)|^2 = a_0^2 T_s \mathrm{sinc}^2\left(\frac{\omega T_s}{2}\right)$$

and the PSD of the modulated signal waveform $s(t)$ is

$$\frac{1}{4}\left[a_0^2 T_s \text{sinc}^2\left(\frac{(\omega-\omega_C)T_s}{2}\right)+a_0^2 T_s \text{sinc}^2\left(\frac{-(\omega+\omega_C)T_s}{2}\right)\right]$$

(d) For BPSK, $T_s = 1/R_b$ since there is 1 bit per symbol. For QPSK, $T_s = 2/R_b$ since there are 2 bits per symbol. For 16-PSK, $T_s = 4/R_b$ since there are 4 bits per symbol.

For all of the modulation schemes, R_{bb} is unchanged, since the symbols remain uncorrelated with each other except when $L = 0$ and they are fully correlated.

Therefore, substituting for T_s: PSD of $p(t)$ for BPSK is $(a_0^2/R_b)\text{sinc}^2(\omega/(2R_b))$ PSD of $p(t)$ for QPSK is $(2a_0^2/R_b)\text{sinc}^2(\omega/R_b)$ PSD of $p(t)$ for 16-PSK is $(4a_0^2/R_b)\text{sinc}^2(2\omega/R_b)$

When the phasors are modulated onto a carrier wave at ω_C, the PSD of $p(t)$ gets shifted up and down by ω_C and scaled by $(1/2)^2 = 1/4$, using the formula in part (b).

Hence the PSDs are sinc^2 functions, shifted to be centred on $\pm\omega_C$ and scaled such that their widths between the first zeros of the spectra are: $4\pi R_b$ for BPSK $2\pi R_b$ for QPSK πR_b for 16-PSK

(e) As we see above the bandwidth is reduced when the number of modulation states is increased, but unfortunately the distance between adjacent states then tends to decrease, which increases the susceptibility to noise and interference and the symbol error rate gets worse. The bandwidth only improves by the log of the number of states whereas for M-PSK the distance between states is inversely proportional to the number of states.

In 16-PSK, the 16 phase states must be uniformly distributed between 0 and 2π. With 16-QAM the states can be uniformly distributed over a square, so the distance between them only decreases as the square root of the number of states. This is a significantly better tradeoff. Sketches of the constellations can be used to demonstrate this (see the lecture notes, fig 5.4 on page 52).

4 (a) Multipath radio transmission occurs when a receiver is able to receive a given wanted signal via more than one transmission path. These paths may have significantly different path delays so that the components of the signal that arrive via different paths may have different arrival times. This means that data bit timing for one path may be be different from that of another path and suboptimal data bit detection will occur. If the bit period is less than the time spread of the path delays then bad intersymbol interference will occur, causing very poor data error rates. For high speed data streams, such as for a digital TV channel operating at several Mbit/s, the tolerance to path delay spread will be very small.

(b) Multiple paths occur mainly due to reflections from large buildings or natural objects such as hills. The biggest path differences tend to arise when signals arrive at the antenna as a result of reflections from widely different directions, whereas a highly directional antenna tends to receive only signals that have come via an almost direct path from the transmitter. Hence the use of an omnidirectional antenna will often result in much larger path delay spreads than a highly directional one. Moving vehicles in general require omni-directional antennas because their direction of travel is uncertain, so they will be subject to greater path delay spreads.

(c) OFDM operates by splitting a single high-speed data stream down into many much lower-speed streams (such that the total data rate is maintained), each modulated onto a separate sub-carrier. Hence the problem of high bit-rates may be overcome. With thousands of subcarriers, as are possible using fast Fourier transform techniques, the symbol rate on each subcarrier may be reduce by 3 or 4 factors of 10. A further trick is to use a guard period that is typically around one quarter of the symbol period on each subcarrier, so that delay spreads of less than the guard period produce no intersymbol interference at all.

A block diagram of an OFDM system is given in fig 6.1 of the lecture notes. The high-speed data stream is demultiplexed into N low-speed streams and these are fed to an inverse FFT block, so that they modulate most of the Fourier coefficients in amplitude and/or phase. The inverse FFT combines all the Fourier components together to produce a composite time-domain signal, comprising many modulated frequency components. The composite signal may be demodulated at the receiver by performing an FFT on it to recover all the separate Fourier coefficients, and from these the data may be extracted by appropriate demodulation. Mutual orthogonality of the Fourier basis functions (sinusoids) ensures that adjacent Fourier components do not cause mutual interference.

(d) Guard periods, as shown in fig 6.2 of the lecture notes, may be added to the analysis periods of the FFTs, in such a way that the symbol transitions, arising from all the differing path delays, can be constrained to lie within the guard periods. This means that every FFT tone is a pure sinusoid for the duration of the analysis period and orthogonality of the tones is maintained, even when they are being modulated.

For the system given: coded bit rate $= 1.2 * 2 = 2.4$ Mb/s. Bit rate per carrier $=$ 2.4 Mb/s$/1600 = 1500$ bit/s. Min symbol rate per carrier with QPSK modulation $(2bit/symbol) = 1500/2 = 750$ sym/s. Max symbol period on each subcarrier $= (1/750)$ s $= 1.333$ ms. Analysis period of DFTs $= 1/(subcarrierspacing) = (1/1000)$ s $= 1$ ms. Max guard period $= 1.333 - 1.0 = 0.333$ ms. Max delay spread $= 0.333$ ms. In distance (using vel of light $= 3.10^8$ m/s) the max delay spread becomes $0.333 \times 10^{-3} \times 3 \times 10^8 =$ 10^5 m $= 100$ km.

(e) In a country the size of the UK, many radio transmitters are needed to ensure that all receivers lie approximately within line-of-sight of a transmitter, because radio waves at VHF only travel in approximately straight lines and the earth is curved. This means that transmitters are typically no more than 80 km apart. For older analogue radio transmissions, each transmitter had to be allocated a frequency that was different from its immediate neighbours so as to avoid unpleasant interfernce effects when receivers were approximately equidistant from two or more transmitters. This typically requires around 7 frequencies to be allocated to each national radio channel. However with OFDM digital transmissions, the signals from adjacent transmitters simply behave like different multipath components, as long as the transmitters are synchronised correctly to each other. Provided the guard periods are designed to tolerate the maximum path delay spread for signals arriving from adjacent transmitters, then only one frequency needs to be allocated to each channel (or group of channels if they are multiplexed together). Thus a 7-fold decrease in bandwith compared to analogue systems can be achieved by this 'trick'. The above system in part (d) with up to 100 km of delay spread, will then easily accommodate transmitters spaced only 80 km apart.

END OF PAPER