Question 1

(a) $\|\boldsymbol{u}\|_1 = 7$, $\|\boldsymbol{u}\|_2 = \sqrt{19}$, $\|\boldsymbol{u}\|_\infty = 4$
$\|\boldsymbol{v}\|_1 = 21$, $\|\boldsymbol{v}\|_2 = \sqrt{185}$, $\|\boldsymbol{v}\|_\infty = 10$
$\|\boldsymbol{w}\|$ same as for $\boldsymbol{v}$

[10%]

(b) (i) $\|\boldsymbol{A}\|_1 = \max_j \sum_i |a_{ij}| = 12$
$\|\boldsymbol{A}\|_2 = \sqrt{\lambda_{\max}(\boldsymbol{A}^T\boldsymbol{A})} = 12$ – obvious given the form of $\boldsymbol{A}$
$\|\boldsymbol{A}\|_\infty = \max_i \sum_j |a_{ij}| = 12$
$\|\boldsymbol{B}\|_1 = \max_j \sum_i |b_{ij}| = 7$
$\|\boldsymbol{B}\|_2 = \sqrt{\lambda_{\max}(\boldsymbol{B}^T\boldsymbol{B})}$
$\boldsymbol{B}^T\boldsymbol{B} = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 25 & -10 \\ -10 & 5 \end{bmatrix}$
$\therefore\ (25-\lambda)(5-\lambda) - 10^2 = 0$
$\therefore\ \lambda^2 - 30\lambda + 25 = 0$
$\therefore\ \lambda_{\max} = 15 + 10\sqrt{2}$
$\therefore\ \|\boldsymbol{B}\|_2 \approx 5.398$
$\|\boldsymbol{B}\|_\infty = \max_i \sum_j |b_{ij}| = 5$

[20%]

(ii) Use the inequality $\|\boldsymbol{AB}\| \le \|\boldsymbol{A}\|\|\boldsymbol{B}\|$
and the triangle inequality $\|\boldsymbol{A}+\boldsymbol{B}\| \le \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$
and noting that $\|\boldsymbol{D}\|_p < 7$:

$\|\boldsymbol{C}+\boldsymbol{D}\|_p < \|\boldsymbol{C}\|_p + \|\boldsymbol{D}\|_p = 12$
$\|\boldsymbol{CD}\|_p < \|\boldsymbol{C}\|_p\|\boldsymbol{D}\|_p = 35$

[10%]

(c) (i) The condition number in the 2-norm, κ_2, is the ratio of the largest singular value over the smallest singular value. The diagonal matrix in a SVD contains the singular values. Therefore $\kappa_2 = 1.030 \times 10^9 / 2.606 \times 10^{-9} \approx 3.95 \times 10^{17}$.

This is an extremely large condition number, and will likely lead to unstable computations.

[15%]

(ii) Introducing an error $\delta\boldsymbol{b}$, we have the perturbed system $\boldsymbol{A}(\boldsymbol{p}+\delta\boldsymbol{p}) = \boldsymbol{b}+\delta\boldsymbol{b}$. Since $\boldsymbol{Ap} = \boldsymbol{b}$ (exact problem), we have $\boldsymbol{A}\delta\boldsymbol{p} = \delta\boldsymbol{b} \implies \delta\boldsymbol{p} = \boldsymbol{A}^{-1}\delta\boldsymbol{b}$.
Taking norms:

$$\|\boldsymbol{Ap}\| = \|\boldsymbol{b}\| \le \|\boldsymbol{A}\|\|\boldsymbol{p}\|$$
$$\|\boldsymbol{A}^{-1}\delta\boldsymbol{b}\| = \|\delta\boldsymbol{p}\| \le \|\boldsymbol{A}^{-1}\|\|\delta\boldsymbol{b}\|$$

From the first line, we have

$$\frac{1}{\|\boldsymbol{p}\|} \le \frac{\|\boldsymbol{A}\|}{\|\boldsymbol{b}\|}$$

and from the second line, we have

$$\|\delta \boldsymbol{p}\| \leq \|\boldsymbol{A}^{-1}\| \|\delta \boldsymbol{b}\|$$

Combining these two inequalities:

$$\frac{\|\delta \boldsymbol{p}\|}{\|\boldsymbol{p}\|} \leq \|\boldsymbol{A}\| \|\boldsymbol{A}^{-1}\| \frac{\|\delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|}$$

[30%]

(iii) Use a change of variable, i.e. use $x^\star = x - 800$ in place of x. This will reduce the condition number dramatically.

[15%]

Assessor's Comments:

86 attempts, Average mark 14.3/20, Maximum 20, Minimum 2.

A popular question, attempted by 89.6% of candidates. Attempts ranged from the perfect to the derisory, but, as indicated by the high average mark, there were many very good attempts.

A surprisingly large number of candidates could not spot the 2-norm of the diagonal matrix $\boldsymbol{A}$ in part (b)(i) by inspection.

In part (c)(i) some candidates recognized that the matrix was ill-conditioned but did not (or could not?) explain what this meant in practical terms.

Several solutions to part (c)(ii) were undermined by lack of clarity/explanation in the proof presented.

In part (c)(iii) very few candidates spotted the significance of the values of x_i all being close to 800 and what this made possible. Most answers discussed (much) more elaborate reformulations than the one expected.

Q2

(a) The total MW-hours generated by CCGT $= E(x_1)$

The total MW-hours generated by diesel $= E(x_1 + x_2) - E(x_1)$

The total MW-hours purchased from the grid $= E(P_{\max}) - E(x_1 + x_2)$

Total cost per annum = capital cost + operational cost + cost of power bought

$$\therefore \quad f(x_1, x_2) = c_1 x_1 + c_2 x_2 + r_1 E(x_1) + r_2\left[E(x_1 + x_2) - E(x_1)\right] + e\left[E(P_{\max}) - E(x_1 + x_2)\right]$$

Constraints are

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_1 + x_2 \leq P_{\max}$$

[15%]

(b)

(i) The first-order necessary condition is

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$$

$$\therefore \quad \frac{\partial f}{\partial x_1} = c_1 + r_1 E'(x_1) + r_2 E'(x_1 + x_2) - r_2 E'(x_1) - eE'(x_1 + x_2) = 0$$

where $E'(x) = \dfrac{dE(x)}{dx}$

$$\therefore \quad \frac{\partial f}{\partial x_1} = c_1 + (r_1 - r_2)E'(x_1) + (r_2 - e)E'(x_1 + x_2) = 0 \qquad (2.1)$$

$$\therefore \quad \frac{\partial f}{\partial x_2} = c_2 + r_2 E'(x_1 + x_2) - eE'(x_1 + x_2) = 0$$

$$\therefore \quad \frac{\partial f}{\partial x_2} = c_2 + (r_2 - e)E'(x_1 + x_2) = 0 \qquad (2.2)$$

(ii) These equations will govern a minimum if the Hessian $\boldsymbol{H}$ is positive definite.

$$\frac{\partial^2 f}{\partial x_1^2} = (r_1 - r_2)E''(x_1) + (r_2 - e)E''(x_1 + x_2)$$

where $E''(x) = \dfrac{d^2 E(x)}{dx^2}$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = (r_2 - e)E''(x_1 + x_2)$$

$$\frac{\partial^2 f}{\partial x_2^2} = (r_2 - e)E''(x_1 + x_2)$$

Let $A = (r_1 - r_2)E''(x_1)$ and $B = (r_2 - e)E''(x_1 + x_2)$. The Hessian can then be written as

$$\boldsymbol{H} = \begin{bmatrix} A+B & B \\ B & B \end{bmatrix}$$

$\boldsymbol{H}$ is positive definite if $\quad A+B>0$

and if $\quad \det(\boldsymbol{H})>0$

$$\therefore \quad (A+B)B-B^2>0 \quad \Rightarrow \quad AB>0$$

$A+B>0$ and $AB>0$ imply that $A>0$ and $B>0$

$$(r_1-r_2)E''(x_1)>0 \text{ and } (r_2-e)E''(x_1+x_2)>0$$ [45%]

(c) Subtracting equation (2.2) from equation (2.1)

$$c_1-c_2+(r_1-r_2)E'(x_1)=0$$

$$\therefore \quad E'(x_1)=\frac{c_1-c_2}{r_2-r_1}$$

For the specified form of $E(x)$

$$E'(x)=\frac{\pi E_0}{2P_{\max}}\cos\left(\frac{\pi x}{2P_{\max}}\right)$$

$$\therefore \quad \frac{\pi E_0}{2P_{\max}}\cos\left(\frac{\pi x_1}{2P_{\max}}\right)=\frac{c_1-c_2}{r_2-r_1}$$

$$\therefore \quad \frac{\pi\times 3.942\times 10^6}{2\times 900}\cos\left(\frac{\pi x_1}{2\times 900}\right)=\frac{90000-45000}{105-75}$$

$$\therefore \quad 6880\cos\left(\frac{\pi x_1}{1800}\right)=1500 \quad \Rightarrow \quad \cos\left(\frac{\pi x_1}{1800}\right)=0.2180$$

$$\therefore \quad \frac{\pi x_1}{1800}=\cos^{-1}(0.2180)=1.351 \quad \Rightarrow \quad x_1=774.1\,\text{MW}$$

Rearranging equation (2.2)

$$E'(x_1+x_2)=\frac{c_2}{e-r_2}$$

$$\therefore \quad \frac{\pi E_0}{2P_{\max}}\cos\left(\frac{\pi(x_1+x_2)}{2P_{\max}}\right)=\frac{c_2}{e-r_2}$$

$$\therefore \quad 6880\cos\left(\frac{\pi(x_1+x_2)}{1800}\right)=\frac{45000}{150-105}=1000 \quad \Rightarrow \quad \cos\left(\frac{\pi(x_1+x_2)}{1800}\right)=0.1453$$

$$\therefore \quad \frac{\pi(x_1+x_2)}{1800}=\cos^{-1}(0.1453)=1.425 \quad \Rightarrow \quad x_1+x_2=816.4\,\text{MW} \quad \Rightarrow \quad x_2=42.3\,\text{MW}$$

The second-order conditions <u>must</u> be checked

$$E''(x)=-\frac{\pi^2 E_0}{4P_{\max}^2}\sin\left(\frac{\pi x}{2P_{\max}}\right)$$

$$\therefore \quad A = (r_1 - r_2)E''(x_1) = -(75 - 105)\frac{\pi^2 \times 3.942 \times 10^6}{4 \times 900^2}\sin\left(\frac{\pi \times 774.1}{2 \times 900}\right)$$

$$\therefore \quad A = 351.6 > 0 \quad \Rightarrow \quad \text{OK}$$

$$\therefore \quad B = (r_2 - e)E''(x_1 + x_2) = -(105 - 150)\frac{\pi^2 \times 3.942 \times 10^6}{4 \times 900^2}\sin\left(\frac{\pi \times 816.4}{2 \times 900}\right)$$

$$\therefore \quad B = 534.6 > 0 \quad \Rightarrow \quad \text{OK}$$

So, the solution is a minimum.

The constraints must also be checked:

$$x_1 = 774.1\,\text{MW} \;\geq\; 0 \quad \Rightarrow \quad \text{OK}$$

$$x_2 = 42.3\,\text{MW} \;\geq\; 0 \quad \Rightarrow \quad \text{OK}$$

$$x_1 + x_2 = 816.4\,\text{MW} \;\leq\; P_{\max} = 900\,\text{MW} \quad \Rightarrow \quad \text{OK}$$

The constraints are all inactive, so the solution is indeed an interior point.

Thus, the optimal balance of generating capacity is $x_1 = 774.1\,\text{MW}$, $x_2 = 42.3\,\text{MW}$. [40%]

Assessor's Comments:

88 attempts, Average mark 14.4/20, Maximum 19, Minimum 5.

The most popular question, attempted by 91.7% of candidates, and by a narrow margin the best done. There were many very good attempts, though no perfect ones.

The most common failings were:

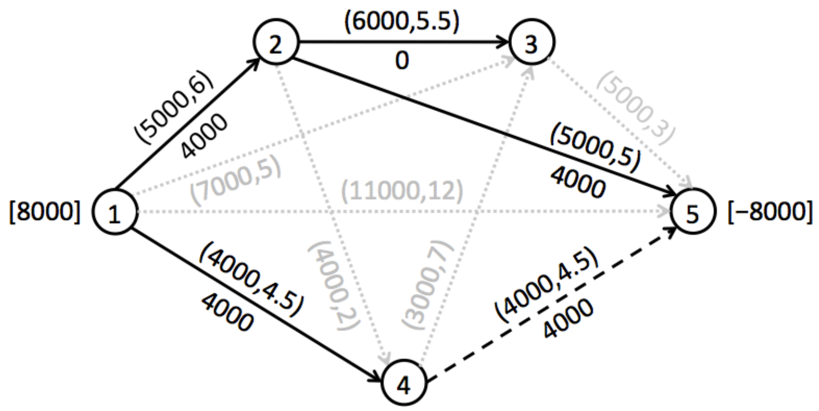
1. Sloppiness/lack of thought in defining the constraints in part (a);
2. The use of incorrect second-order conditions in part (b)(ii);
3. Lack of clarity in arguments/explanations in part (b)(ii);
4. Overly complicated attempts to part (c) that almost inevitably went astray;
5. Failure to check the second-order conditions and/or the constraints, having found the optimal balance of generating capacity in part (c).

Q3

(a) Fig. 2 as drawn does not represent a spanning tree, because node 3 is disconnected from the rest of the network and the arcs in the basis form a loop.

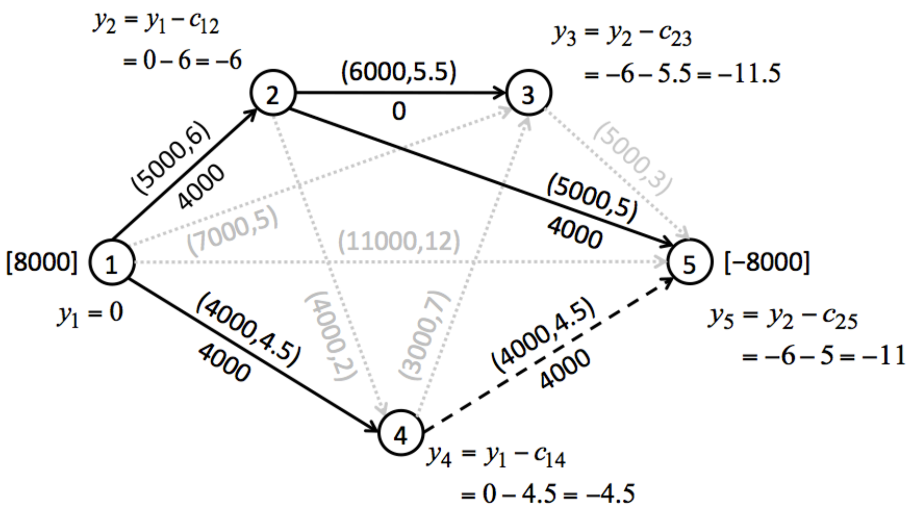
To form a feasible basic solution (spanning tree) node 3 must be connected to another node (say node 2) by a zero flow arc in the basis, and one of the arcs with flow at its upper limit must be made non-basic in order to break the loop.

A possible feasible basis is therefore:



[15%]

(b) Taking $y_1 = 0$, the remaining simplex multipliers for the basis above are:



The corresponding reduced costs for the non-basic variables are:

LB: $\bar{c}_{13} = c_{13} - y_1 + y_3 = 5 - 0 + (-11.5) = -6.5$

LB: $\bar{c}_{15} = c_{15} - y_1 + y_5 = 12 - 0 + (-11) = 1$

LB: $\bar{c}_{24} = c_{24} - y_2 + y_4 = 2 - (-6) + (-4.5) = 3.5$

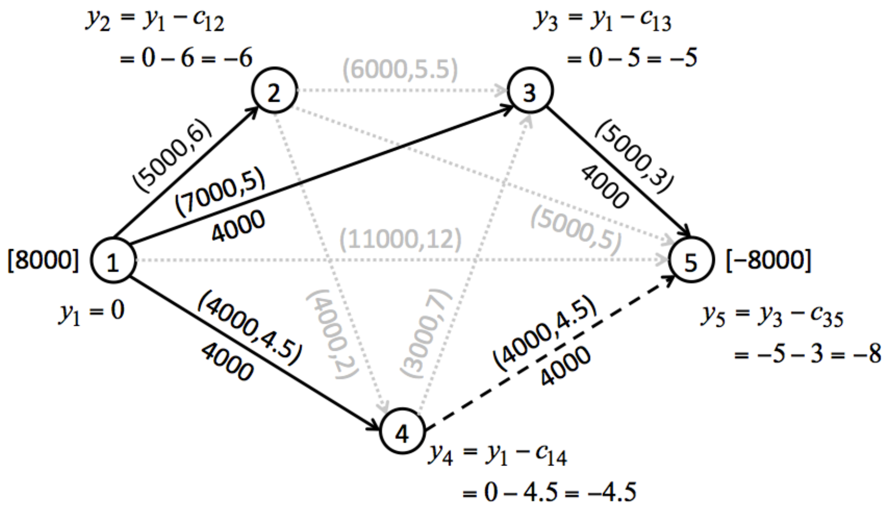
LB: $\bar{c}_{35} = c_{35} - y_3 + y_5 = 3 - (-11.5) + (-11) = 3.5$

LB: $\bar{c}_{43} = c_{43} - y_4 + y_3 = 7 - (-4.5) + (-11.5) = 0$

UB: $\bar{c}_{45} = c_{45} - y_4 + y_5 = 4.5 - (-4.5) + (-11) = -2$

As arc 1–3, which is at its lower bound, has a negative reduced cost, this cannot be the optimal allocation. [30%]

(c) Taking $y_1 = 0$ again, the remaining simplex multipliers for the Fig. 3 basis are:



The corresponding reduced costs for the non-basic variables are:

LB: $\bar{c}_{15} = c_{15} - y_1 + y_5 = 12 - 0 + (-8) = 4$

LB: $\bar{c}_{23} = c_{23} - y_2 + y_3 = 5.5 - (-6) + (-5) = 6.5$

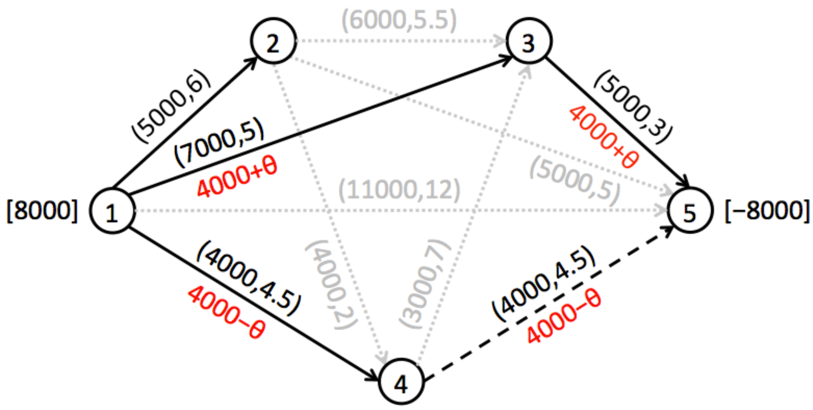
LB: $\bar{c}_{24} = c_{24} - y_2 + y_4 = 2 - (-6) + (-4.5) = 3.5$

LB: $\bar{c}_{25} = c_{25} - y_2 + y_5 = 5 - (-6) + (-8) = 3$

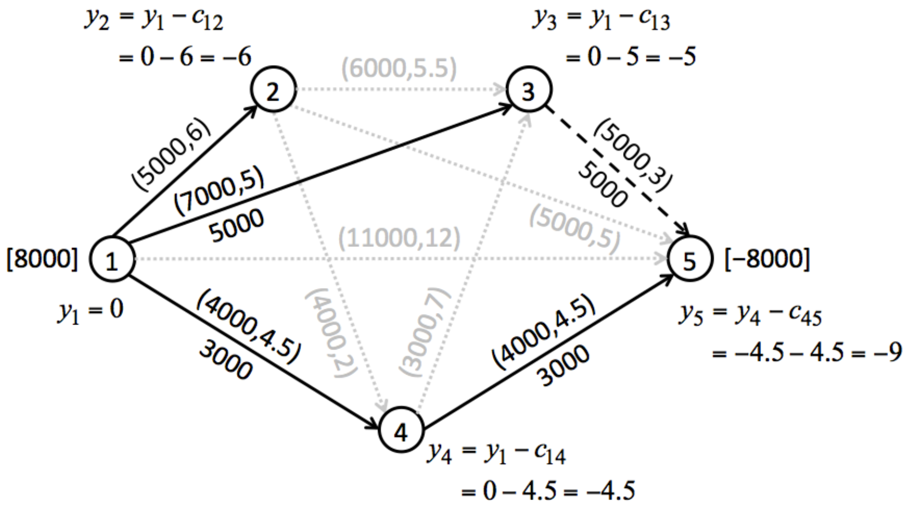
LB: $\bar{c}_{43} = c_{43} - y_4 + y_3 = 7 - (-4.5) + (-5) = 6.5$

UB: $\bar{c}_{45} = c_{45} - y_4 + y_5 = 4.5 - (-4.5) + (-8) = 1$

The lower bound arcs all have positive reduced costs, but the upper bound arc 4–5 also has a positive reduced cost, so the network cost can be decreased by reducing the flow on it:



θ is limited to 1000 by the capacity of arc 3–5. Arcs 1–4 and 4–5 now have flow below their capacity and therefore must be in the basis, as must arc 1–3. Therefore arc 3–5 drops out of the basis. The new basis and associated simplex multipliers are then:



The corresponding reduced costs for the non-basic variables are:

LB: $\bar{c}_{15} = c_{15} - y_1 + y_5 = 12 - 0 + (-9) = 3$

LB: $\bar{c}_{23} = c_{23} - y_2 + y_3 = 5.5 - (-6) + (-5) = 6.5$

LB: $\bar{c}_{24} = c_{24} - y_2 + y_4 = 2 - (-6) + (-4.5) = 3.5$

LB: $\bar{c}_{25} = c_{25} - y_2 + y_5 = 5 - (-6) + (-9) = 2$

LB: $\bar{c}_{43} = c_{43} - y_4 + y_3 = 7 - (-4.5) + (-5) = 6.5$

UB: $\bar{c}_{35} = c_{35} - y_3 + y_5 = 3 - (-5) + (-9) = -1$

The lower bound arcs all have positive reduced costs, and the one upper bound arc has a negative reduced cost, so this is the optimal flow allocation. [55%]

Assessor's Comments:

83 attempts, Average mark 13.4/20, Maximum 20, Minimum 1.
A fairly straightforward network flow optimization problem, but the first time this topic has featured in a 3M1 exam.

There were many very good answers but the average mark was dragged down by a tail of attempts from candidates who had clearly not anticipated a question on this topic coming up.

In part (a), many recognized that the basis proposed was not a spanning tree, but failed explicitly to mention the need to break the loop.

Many candidates suggested a different initial set of flows, rather than just modifying the basis for the given set of flows. This different reading of the question was not penalized.

Some candidates confused the standard method for finding an initial feasible flow solution (by defining an artificial problem with additional arcs) and the task of finding a feasible basis for a defined (feasible) flow solution.

In part (b), several answers were undermined by the candidate's failure to specify exactly what basis was being used. Some candidates used incorrect formulae for the simplex multipliers and reduced costs despite these being given in the question. Some candidates attempted to analyse a basis including a loop, resulting in inconsistent multiplier values, and then (incorrectly) took this as evidence that the solution was not optimal.

The most common mistakes in part (c) arose from lack of understanding of the rules determining whether flows are part of the basis and from incorrect interpretation of reduced cost values.

4.

(a) Set $p(x,t) = X(x)T(t)$. Substituting in yields the following pair of differential equations:

$$\begin{aligned} \frac{\partial T(t)}{\partial t} + \alpha k^2 T(t) &= 0 \\ \frac{\partial^2 X(x)}{\partial x^2} + k^2 X(x) &= 0 \end{aligned}$$

The solutions are then:

$$\begin{aligned} T(t) &\propto \exp(-\alpha k^2 t) \\ X(x) &\propto \exp(ikx) \end{aligned}$$

Thus, the general solution is

$$p(x,t) = a(k)\exp(-\alpha k^2 t)\exp(ikx)$$

This is satisfied by any k, so the general solution is

$$p(x,t) = \int_{-\infty}^{\infty} a(k)\exp(-\alpha k^2 t)\exp(ikx)dk$$

[25%]

(b)(i) [derivation in lecture notes]
Need to satisfy the initial condition at $t = 0$, hence

$$\delta(x) = p(x,0) = \int_{-\infty}^{\infty} a(k)\exp(ikx)dk = \int_{-\infty}^{\infty} a(-\tilde{k})\exp(-i\tilde{k}x)d\tilde{k}$$

By noting that this is the Fourier Transform $\mathcal{F}()$ of $a(-\tilde{k})$

$$a(-\tilde{k}) = \mathcal{F}^{-1}\{\delta(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)\exp(i\tilde{k}x)dx = \frac{1}{2\pi}$$

Thus, the final solution is

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\alpha k^2 t)\exp(ikx)dk$$

Using the equality given in the question, this can be simplified to the form:

$$p(x,t) = \frac{1}{\sqrt{4\alpha\pi t}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$

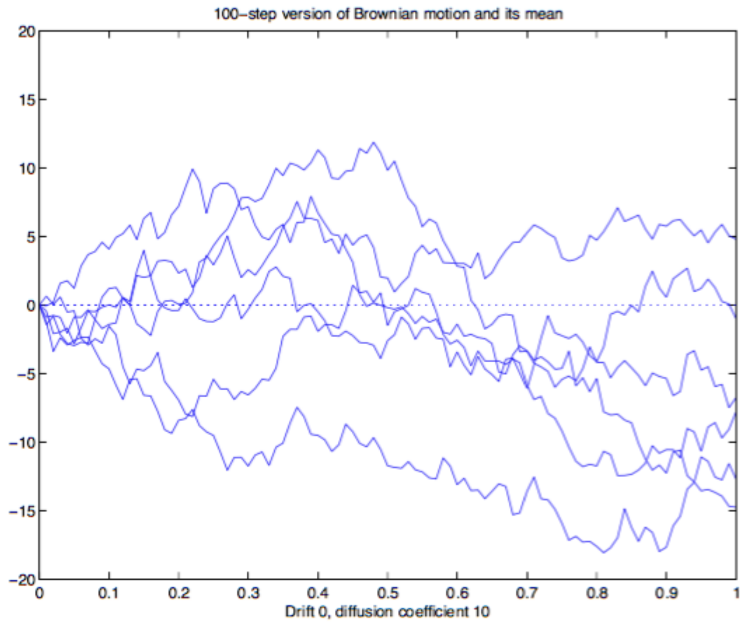
This is a Gaussian distribution with zero mean and variance

$$b(t) = 2\alpha t$$

[35%]

(b)(ii) As discussed in the lecture notes, over time the variance of the distribution of particles increases *linearly*. Examples of possible paths for a single particle are:

[15%]



(c) We are interested in the correlation of the particles at time t_1 and time t_2. The denominator terms are known

$$\begin{aligned} \mathcal{E}\{(x_1 - \mu_1)^2\} &= 2\alpha t_1 \\ \mathcal{E}\{(x_2 - \mu_2)^2\} &= 2\alpha t_2 \end{aligned}$$

For the numerator term, we need to rely on the fact that Brownian motion is a Markov process. Thus, the position at time t_2, x_2 can be written as

$$x_2 = x_1 + (x_2 - x_1)$$

The distribution for $(x_2 - x_1)$ will be zero mean and independent of x_1. Thus, noting that $\mu_1 = \mu_2 = 0$,

$$\mathcal{E}\{(x_1 - \mu_1)(x_2 - \mu_2)\} = \mathcal{E}\{x_1 x_2\} = \mathcal{E}\{x_1[x_1 + (x_2 - x_1)]\} = \mathcal{E}\{x_1 x_1\} = 2\alpha t_1$$

Hence

$$\texttt{Corr}(x_1, x_2) = \frac{2\alpha t_1}{2\alpha\sqrt{t_1 t_2}} = \sqrt{\frac{t_1}{t_2}}$$

[25%]

Assessor's Comments:

32 attempts, Average mark 9.2/20, Maximum 20, Minimum 1.

The least popular question by some margin and the least well attempted. Clearly many candidates were not expecting a question on Brownian motion and were ill prepared to attempt it. The average mark was dragged down by the high proportion of desultory attempts, done seemingly out of desperation.

There were a handful of very good answers from candidates who had evidently revised thoroughly rather than tactically.

Among those who made serious attempts at answering the question, lack of detail and careful justification in the various proofs required was the most common cause of lost marks.

Some answers lost marks due to the failure of the candidate to recognize that the problem specified was one-dimensional spatially.