3M1

1. a) $A_{ij} = \bar{A}_{ji} = A_{ji}$ $\therefore$ $\underline{A}$ is symmetric and all entries are real.

b) i $\underline{\underline{Q}}^H \underline{\underline{Q}} = \underline{\underline{I}} \Rightarrow \det(\underline{\underline{Q}}^H \underline{\underline{Q}}) = \det(\underline{\underline{Q}}^H)\det(\underline{\underline{Q}})$

$$= \overline{\det(\underline{\underline{Q}})}\det(\underline{\underline{Q}}) = 1$$

$\therefore$ $|\det(Q)| = 1$ (modulus 1, det can be complex)

ii $\|\underline{\underline{Q}}\,\underline{x}\|_2^2 = \underline{x}^H \underbrace{\underline{\underline{Q}}^H \underline{\underline{Q}}}_{\underline{\underline{I}}} \underline{x} = \underline{x}^H \underline{x} = \|\underline{x}\|_2^2$ (1)

iii Approach 1

$$\|\underline{\underline{A}}\|_2 = \max_{\underline{x} \neq 0} \frac{\|\underline{\underline{A}}\,\underline{x}\|_2}{\|\underline{x}\|_2}$$

$$\|\underline{\underline{Q}} A\|_2 = \max \frac{\|\underline{\underline{Q}}(\underline{\underline{A}}\,\underline{x})\|_2}{\|\underline{x}\|_2} = \max \frac{\|\underline{\underline{A}}\,\underline{x}\|_2}{\|\underline{x}\|_2}$$

(using (1))

$$\|\underline{\underline{A}}\,\underline{\underline{Q}}\|_2 = \max \frac{\|\underline{\underline{A}}\,\underline{\underline{Q}}\,\underline{x}\|_2}{\|\underline{x}\|_2} = \max \frac{\|\underline{\underline{A}}\,\underline{\underline{Q}}\,\underline{x}\|_2}{\|\underline{\underline{Q}}\,\underline{x}\|_2}$$

$$= \max \frac{\|\underline{\underline{A}}\,\underline{y}\|_2}{\|\underline{y}\|_2}$$

$$\Rightarrow \|\underline{\underline{Q}} A\|_2 = \|\underline{\underline{A}}\,\underline{\underline{Q}}\|_2 = \|\underline{\underline{A}}\|_2$$

Approach 2 (using defn of 2-norm in terms of singular values)

- $\|\underline{\underline{Q}}\,\underline{\underline{A}}\|_2^2 = \lambda_{max}((\underline{\underline{Q}}\,\underline{\underline{A}})^H \underline{\underline{Q}}\,\underline{\underline{A}})$

$$= \lambda_{max}(\underline{\underline{A}}^H \underline{\underline{Q}}^H \underline{\underline{Q}}\,\underline{\underline{A}}) = \lambda_{max}(\underline{\underline{A}}^H \underline{\underline{A}}) = \|A\|_2^2$$

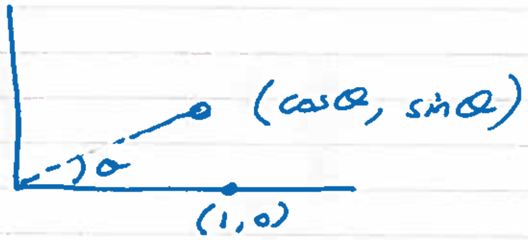
- $\|\underline{\underline{A}}\,\underline{\underline{Q}}\|_2^2 = \lambda_{max}((\underline{\underline{A}}\,\underline{\underline{Q}})^H(\underline{\underline{A}}\,\underline{\underline{Q}}))$

$$= \lambda_{max}(\underline{\underline{A}}\,\underline{\underline{Q}}(\underline{\underline{A}}\,\underline{\underline{Q}})^H)$$ [since $\underline{\underline{C}}\,\underline{\underline{D}}$ and $\underline{\underline{D}}\,\underline{\underline{C}}$ have same eigenvalues]

$$= \lambda_{max}(\underline{\underline{A}}\,\underline{\underline{A}}^H)$$

$$= \lambda_{max}(\underline{\underline{A}}^H \underline{\underline{A}}) = \|\underline{\underline{A}}\|_2^2$$

iv No. Counter example:



l_1 and l_∞ norms change when rotated through angle θ.

4) i) $\underline{x}^H \underline{x} = \|\underline{x}\|_2^2 \rightarrow$ real

$\underline{x}^H \underline{\underline{M}}\, \underline{x} = (\underline{x}^H \underline{\underline{M}}\, \underline{x})^H = \underline{x}^H \underline{\underline{M}}^H \underline{x} = \underline{x}^H \underline{\underline{M}}\, \underline{x}$
(since $\underline{\underline{M}} = \underline{\underline{M}}^H$)
$\rightarrow$ real

$\therefore R(\underline{\underline{M}}, \underline{x})$ must be real.

ii) $$R(\underline{\underline{M}}, c\underline{x}) = \frac{(\bar{c}\underline{x}^H)\underline{\underline{M}}\, c\underline{x}}{(\bar{c}\underline{x}^H)(c\underline{x})} = \frac{|c|^2}{|c|^2}\frac{\underline{x}^H \underline{\underline{M}}\, \underline{x}}{\underline{x}^H \underline{x}}$$
$$= R(\underline{\underline{M}}, \underline{x})$$

iii Eigenvectors of Hermitian matrix are orthogonal, and eigenvalues are real.

$$\underline{x} = \sum_i \alpha_i \underbrace{\underline{u}_i}_{\text{ith eigenvector}}$$

$$R(\underline{\underline{M}}, \underline{x}) = \frac{\left(\sum_i \bar{\alpha}_i \underline{u}_i^H\right) \underline{\underline{M}} \left(\sum_j \alpha_j \underline{u}_j\right)}{\left(\sum_i \bar{\alpha}_i \underline{u}_i^H\right)\left(\sum_j \alpha_j \underline{u}_j\right)}$$

$$= \frac{\sum_i |\alpha_i|^2 \lambda_i}{\sum_i |\alpha_i|^2}$$

This is a weighted average of the eigenvalues, hence maximum is λ_{max} and minimum is λ_{min}

iv $R(\underline{\underline{M}}, \underline{x})$ is maximum when $\underline{x}$ is the eigenvector associated with λ_{max}.

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2. *Markov Chains and Stationary Distributions*

(a) Transition matrix is:

$$\mathbf{P} = \begin{bmatrix} 0.75 & 0.25 & 0.0 \\ 0.25 & 0.0 & 0.75 \\ b & 0.75 & a \end{bmatrix}$$

Constraints on a and b are:

$$a \geq 0; \quad b \geq 0$$
$$a + b + 0.75 = 1.0$$

[15%]

(b) For a stationary distribution

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$$

By inspection for the stationary distribution to be all equal, as shown, then each column of $\mathbf{P}$ must sum to 1.0. Thus

$$a = 0.25; \quad b = 0.0$$

[20%]

(c)(i) As $\pi_j = \pi_k$ for all three states, it is sufficient to show that the $\mathbf{P}$ is symmetric. For the values of a and b in Part (b) this is true. Hence the process is in detailed balance. [10%]

(c)(ii) From lecture notes

$$(\boldsymbol{\pi}\mathbf{P})_k = \sum_j \pi_j p_{j,k} = \sum_j \pi_k p_{k,j} = \pi_k$$

[10%]

(d)(i) Assume that the transition is from state j to k ($k \neq j$). Using the proposal function the probability of this transition is $r_{j,k}$ However, this point is only kept with probability α given in the question. Thus the equivalent transition is

$$\overline{r}_{j,k} = \alpha r_{j,k} = r_{j,k} \text{min}\left\{\frac{\pi_k r_{k,j}}{\pi_j r_{j,k}}, 1\right\}$$

as required. For the self loop there is the sum to one constraint. Yielding the expression required in the question. [15%]

(d)(ii) From the lecture notes

$$\begin{aligned}
\pi_j \overline{r}_{j,k} &= \pi_j r_{j,k} \min\left\{\frac{\pi_k r_{k,j}}{\pi_j r_{j,k}}, 1\right\} \\
&= \min\left\{\pi_k r_{k,j}, \pi_j r_{j,k}\right\} \\
&= \pi_k r_{k,j} \min\left\{1, \frac{\pi_j r_{j,k}}{\pi_k r_{k,j}}\right\} = \pi_k \overline{r}_{k,j}
\end{aligned}$$

Hence the process is in detailed balance, so from Part (c) $\boldsymbol{\pi}$ is a stationary distribution of this process. [20%]

(d)(iii) No. Any symmetric distribution will have the stationary distribution from Part (b). The exact process will depend on the proposal process R. [10%]

3. *Optimisation*

(a) A set of nested ellipses and a line

(b) The penalised problem is to minimize

$$P = x^2 + 2y^2 + \mu(x + 2y - 3)^2$$

Partial derivatives of P with respect to x and y are

$$\begin{aligned}
\frac{\partial P}{\partial x} &= 2x + 2\mu(x + 2y - 3) \\
\frac{\partial P}{\partial y} &= 4y + 4\mu(x + 2y - 3)
\end{aligned}$$

Setting these to zero gives

$$\begin{aligned}
3\mu &= (1 + \mu)x + 2\mu y \\
3\mu &= \mu x + (1 + 2\mu)y
\end{aligned}$$

Solving by variable substitution,

$$(1 + \mu)(3\mu - (1 + 2\mu)y) + 2\mu^2 y = 3\mu^2$$

$$\begin{aligned}
y &= \frac{3\mu}{(1 + \mu)(1 + 2\mu) - 2\mu^2} = \frac{3\mu}{1 + 3\mu} \\
x &= \frac{1}{\mu}\left(3\mu - (1 + 2\mu)\frac{3\mu}{1 + 3\mu}\right) = \frac{3\mu}{1 + 3\mu}
\end{aligned}$$

The locus of the penalised solution is a line segment ending at $(x, y) = (1, 1)$ with the solution tending to the end as $\mu \to \infty$.

(c) Introducting the Lagrange multiplier, the minimisation problem is now

$$L = x^2 + 2y^2 + \lambda(x + 2y - 3)$$

The equations to be solved are

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2x + \lambda \\ \frac{\partial L}{\partial y} &= 4y + 2\lambda \\ 3 &= x + 2y \end{aligned}$$

which have the solution $x = y = 1$, $\lambda = -2$. The value of lambda is ratio between the gradient of the original function and the gradient of the constraint at the solution.