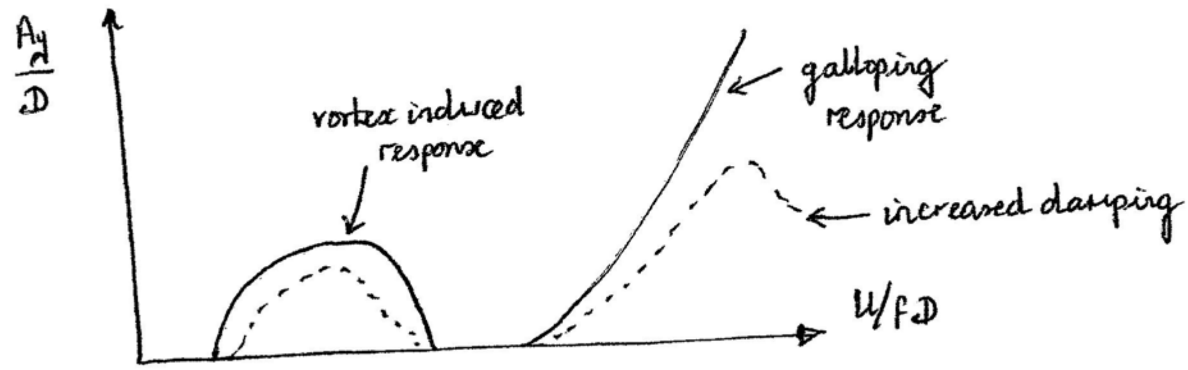
1 (a) Mechanisms are called 'vortex-induced vibration' & 'galloping'.

<u>Galloping</u> requires a lift coefficient that varies with the angle of attack (so a non circular section is req$^d$)

<u>Vortex-induced vibration</u> is due to vortex shedding - any bluff body can shed vortices.

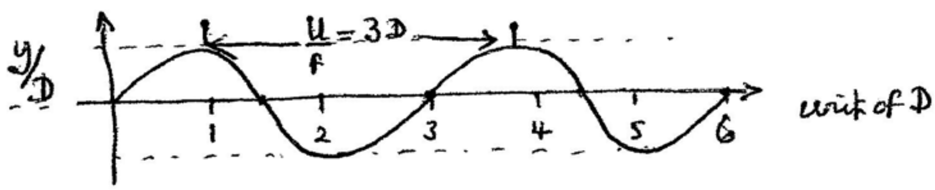



Response could be damped by, eg.
- increasing mass of structure*
- increasing stiffness
- avoiding resonance with low vel. ratio

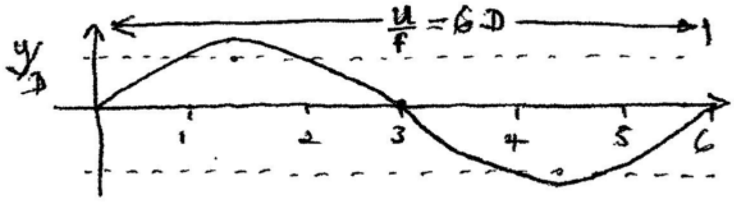
* external damping devices, e.g., (eg. stockbridge damper on power lines)

(b) $\frac{U}{fD}$ is a dimensionless flow velocity. It may be regarded as the ratio of the flow velocity $U$ to a characteristic transverse velocity $fD$. Also note $\frac{U}{fD}$ is the path length per cycle (wavelength of motion) relative to physical scale $D$ of the structure.

(i) $\frac{U}{fD} = 3$




(ii) $\frac{U}{fD} = 6$



1

(c) Added mass refers to the mass of fluid accelerated as a consequence of the motion of the structure.

The mass ratio is defined as the structure's mass $m$ relative to the fluid mass mobilised by the motion of the structure, i.e. $\frac{m}{\rho V}$, $V$ being volume mobilised.

If mass ratio small $\left(\frac{m}{\rho V} \ll 1\right)$, as for a ping-pong ball rising in water, neglecting the added mass term would result in overprediction of ball's rise velocity — as mass of fluid to be accelerated 'out of the way' of the ball would far exceed the mass of the ball itself.

(d) Given
$$\frac{y}{D} = \frac{\frac{1}{2}\rho U^2 C_L \sin(\omega_s t + \phi)}{k\sqrt{[1-(\omega_s/\omega_y)^2]^2 + (2\zeta\omega_s/\omega_y)^2}} = \frac{A_y}{D}\sin(\omega_s t + \phi)$$

Resonant amplitude occurs when shedding frequency equals natural frequency

$$\text{i.e.} \quad \left.\frac{A_y}{D}\right|_{\omega_s=\omega_y} = \left.\frac{\frac{1}{2}\rho U^2 C_L}{k\sqrt{(1-(\omega_s/\omega_y)^2)^2+(2\zeta\omega_s/\omega_y)^2}}\right|_{\omega_s=\omega_y}$$

$\therefore$ Resonant amplitude of vibration is
$$\left.\frac{A_y}{D}\right|_{\omega_s=\omega_y} = \frac{\rho U^2 C_L}{4\zeta}$$

(d) To deduce that this resonant amplitude of vibration is independent of the flow velocity, substitute for the Strouhal no. $S = \frac{f_s D}{U}$ (where $f_s = (2\pi)^{-1}\omega_s$ is shedding frequency) and the reduced damping $\delta_r$

PTO.

1

(d)

We found $\left.\frac{A_y}{D}\right|_{\omega_s=\omega_y} = \frac{\rho U^2 C_L}{4k\zeta}$ — (1)

Introduce $S = \frac{f_s D}{U}$ — (2) Strouhal no.

$\delta_r = 2m \frac{2\pi\zeta}{\rho D^2}$ — (3) Reduced damping

Sub. $U^2 = \left(\frac{f_s D}{S}\right)^2$ from (2) into (1)

$$\left.\frac{A_y}{D}\right|_{\omega_s=\omega_y} = \frac{\rho C_L}{4k\zeta} \cdot \frac{f_s^2 D^2}{S^2}$$

$$\times \frac{\delta_r}{\delta_r} \qquad = \frac{C_L}{S^2 \delta_r} \frac{f_s^2 \pi}{k/m}$$

Now $\omega_y = 2\pi f_y = \sqrt{k/m}$ is natural freq. of structure, hence as $\frac{k}{m} = (2\pi)^2 f_y^2$

$$\left.\frac{A_y}{D}\right|_{f_s=f_y} = \frac{C_L}{S^2 \delta_r} \frac{f_s^2 \pi}{(2\pi)^2 f_y^2}$$

We are concerned with resonance response (for which $f_s = f_y$), hence

$$\left.\frac{A_y}{D}\right|_{f_s=f_y} = \frac{C_L}{4\pi S^2 \delta_r}$$

No dependence here on flow velocity.

As Strouhal no. is constant (approx.) for a range of Re's (Strouhal no. fixes relationship between cylinder & fluid frequencies) $\left.\frac{A_y}{D}\right|_{\omega_s=\omega_y}$ is indep. of flow velocity.

**Q1: Assessor's comments**

Not a very popular question. This question focused on vortex-induced vibration of a structure. Surprisingly, students performed less well on the descriptive, than on the analytical, aspects of the question.

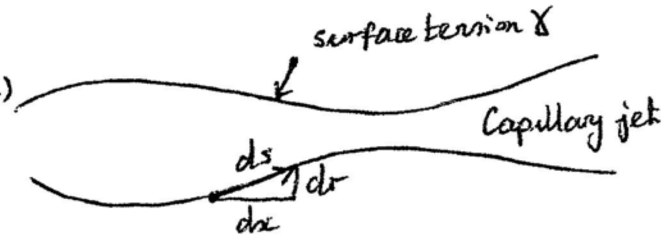
2 (a). Given $r = \alpha + \beta \cos kx$

Use vol. cons. to relate $\alpha$ to $\beta$: $\underbrace{\pi a^2 \lambda}_{\text{vol. undisturbed jet in length } \lambda} = \int_0^\lambda \pi r^2 dx = \underbrace{\pi \int_0^\lambda (\alpha + \beta \cos kx)^2 dx}_{\text{vol. perturbed jet in length } \lambda}$

$$\therefore \quad \frac{\pi}{\pi} a^2 \lambda = \int_0^\lambda \alpha^2 + 2\alpha\beta \cos kx + \beta^2 \cos^2 kx \, dx.$$

$$\Rightarrow \quad a^2 = \alpha^2 + \frac{\beta^2}{2} \quad , \text{ i.e. } \alpha = a\left(1 - \frac{1}{2}\frac{\beta^2}{a^2}\right)^{1/2} . \qquad -(1)$$

$PE_{jet} = \gamma \times (\text{surface area}) \quad -(2)$




Surface area disturbed jet in length $\lambda$

$$SA_1 = \int_{x=0}^{\lambda} 2\pi r \, ds \qquad -(3)$$

Given $ds^2 = dx^2 + dr^2 = dx^2\left(1 + \frac{dr^2}{dx^2}\right) \Rightarrow ds = dx\left(1 + \left(\frac{dr}{dx}\right)^2\right)^{1/2}$

i.e. $ds = dx\left(1 + \frac{1}{2}\left(\frac{dr}{dx}\right)^2 + \ldots\right)$ and $\frac{dr}{dx} = -\beta k \sin kx$

so that $ds \approx \left(1 + \frac{1}{2}\beta^2 k^2 \sin^2 kx\right) dx$

Hence,

$$SA_1 = 2\pi \int_0^\lambda (\alpha + \beta \cos kx)\left(1 + \frac{1}{2}\beta^2 k^2 \sin^2 kx\right) dx$$

$$SA_1 = 2\pi\alpha\lambda\left[1 + \left(\frac{\beta k}{2}\right)^2\right]$$

$$\Rightarrow \quad PE_{disturb\,jet} = \gamma \, 2\pi\alpha\lambda\left(1 + \left(\frac{\beta k}{2}\right)^2\right)$$

Surface area undisturbed jet in length $\lambda$ $= 2\pi a \lambda$

$$\Rightarrow \quad PE_{undisturbed\,jet} = \gamma \, 2\pi a \lambda$$

Hence, $\dfrac{PE_{disturbed}}{PE_{undisturbed}} = \dfrac{\alpha}{a}\left[1 + \dfrac{\beta^2 k^2}{4}\right]$. Now eliminate $\alpha$. $\quad -(4)$

2(a) cont$^{d}$.

Now $\alpha = a\left(1 - \frac{1}{2}\frac{\beta^2}{a^2}\right)^{1/2}$

$= a\left(1 - \frac{1}{2}\frac{1}{2}\frac{\beta^2}{a^2} + \dots\right)$ , $\left|\frac{1}{2}\frac{\beta^2}{a^2}\right| < 1$

So for small amplitude disturbances $\alpha \approx a\left(1 - \frac{1}{4}\frac{\beta^2}{a^2}\right)$ —(5)

Sub. for (5) into (4)

$$\frac{PE_{\text{disturbed}}}{PE_{\text{undisturbed}}} = \left(1 - \frac{1}{4}\frac{\beta^2}{a^2}\right)\left(1 + \frac{\beta^2 k^2}{4}\right)$$

$$= \left(1 - \frac{1}{4}\frac{\beta^2}{a^2}\right)\left(1 + \frac{\beta^2 k^2 a^2}{4a^2}\right) \qquad \times \frac{a^2}{a^2}$$

$$= 1 + \frac{1}{4}\frac{\beta^2}{a^2}k^2a^2 - \frac{1}{4}\frac{\beta^2}{a^2} - \underbrace{\frac{1}{4}\frac{1}{4}\left(\frac{\beta^2}{a^2}\right)^2 k^2 a^2}_{\text{neglect as product of small quantity}}$$

$$= \underline{1 + \frac{1}{4}\frac{\beta^2}{a^2}\left(k^2a^2 - 1\right)} \quad \square$$

For dimensionless wavenumbers $\underline{ka > 1}$, surface area increased, PE increased STABLE.

— " — $\underline{ka < 1}$, surface area decreased, PE decreased UNSTABLE

Given $\lambda = \frac{2\pi}{k}$ and $ka < 1$ for instability

$\frac{2\pi}{\lambda}a < 1$ — " —

ie $\underline{\lambda > 2\pi a \quad \text{for instability}}$ ie wavelength of disturbances exceeding jets initial circumference spontaneously grow.

2(b) We expect growth rate $S = F(k, a, \gamma, \rho)$

& that dimensionless growth rate is a function of the dimensionless wavenumber $ka$.

Dimensions of individual quantities are

$$S \sim [1/T]$$
$$k \sim [1/L]$$
$$a \sim [L]$$
$$\gamma \sim [MLT^{-2}/L] = [MT^{-2}]$$
$$\rho \sim [ML^{-3}]$$

To nondimensionalise $S$ we multiply by quantity with dimensions of time.
Only the quantity $\gamma$ includes a time dependence

$$\gamma \sim \left[\frac{M}{T^2}\right] \quad \therefore \quad \gamma^{1/2} \sim \left[\frac{M^{1/2}}{T}\right]$$

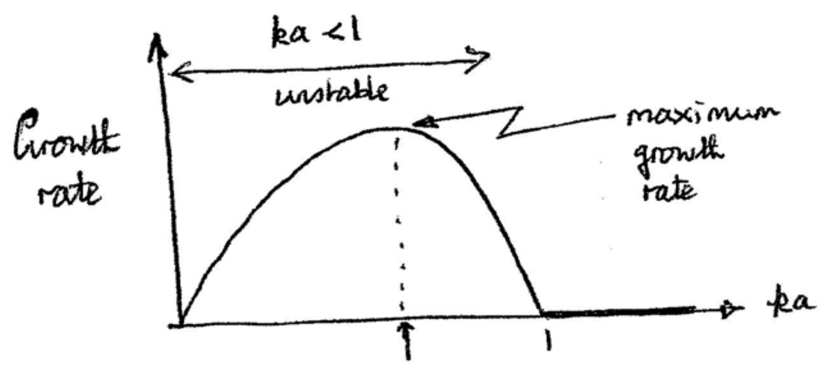
Hence $\quad S\gamma^{-1/2} \sim \left[\frac{1}{T}\right]\left[\frac{M^{1/2}}{T}\right]^{-1} = \left[\frac{1}{M^{1/2}}\right]$

Now $\quad \rho^{1/2} \sim \left[\frac{M^{1/2}}{L^{3/2}}\right]$, hence $\quad \frac{S}{\gamma^{1/2}}\rho^{1/2} \sim \left[\frac{1}{L^{3/2}}\right]$

So finally, $\quad S\,\frac{\rho^{1/2}a^{3/2}}{\gamma^{1/2}}$ is dimensionless growth rate &

$$S\left(\frac{\rho^{1/2}a^{3/2}}{\gamma^{1/2}}\right) = F(ka)$$

2 (c).




Significance of wavenumber at maximal growth rate is that these will be the disturbances observed at small time — i.e. that determines physical scale of the disturbances observed at small time.

**Q2: Assessor's comments**
A popular question on the use of energy arguments in the study of the stability of a capillary jet that was tackled well by most. Students performed, in general, less well where required to provide physical explanations to clarify workings – on the whole, no explanations were offered.

Question 3

(a) Consider a ring of fluid that is displaced outwards to
- radius $r_1$, circumferential velocity $u_1$
- radius $r_2$, with circumferential velocity $u_2'$

Neglecting viscous forces $r_1 u_1 = r_2 u_2'$ as <u>angular mom. conserved</u>

$$\Rightarrow u_2' = \left(\frac{r_1}{r_2}\right) u_1$$

As (Given) $-\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{u_\theta^2}{r}$, the pressure gradient is just sufficient to hold a ring with velocity $u_2$ at the radius $r_2$, thus

if $\frac{u_2'^2}{r_2^2} > \frac{u_2^2}{r_2^2}$, i.e. $u_2'^2 > u_2^2$ then radial press. grad. is not sufficient to offset the centrifugal force & ring continues outwards (unstable)

Thus require $u_2'^2 \leq u_2^2$ for stability.

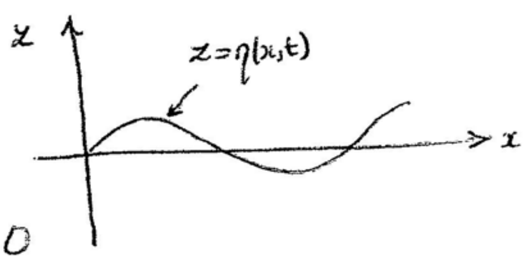
Sub. for $u_2' = (r_1/r_2) u_1$ gives

$$r_1^2 u_1^2 \leq r_2^2 u_2^2 \Rightarrow r_2^2 u_2^2 - r_1^2 u_1^2 \geq 0 \qquad (r_1 > r_2)$$

$$\text{i.e.} \quad \frac{d}{dr}(r^2 u^2) \geq 0$$

now $u = r\Omega$ $\Rightarrow$ $\underline{\frac{d}{dr}(r^2\Omega)^2 \geq 0}$ as req$^d$.

(b). Two-dimensional mixing layer:




For inviscid flow $\nabla \cdot \underset{\sim}{u} = 0$.

Assume irrotational $\underset{\sim}{u} = \nabla\phi$ as $\nabla \times \underset{\sim}{u} = 0$

$\Rightarrow \nabla^2\phi = 0$ <u>Governing eq$^n$.</u>

Question 3 cont$^{d}$.

(b) Kinematic b.c:
Particles on interface, remain on interface. So defining $F(x,y,t) = z - \eta(x,t)$
then $F=0$ on $z=\eta$ and $DF/Dt = 0$ gives

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w \qquad \text{on } z=\eta$$

Dynamic b.c:
Pressure is continuous across interface/vortex sheet, so from unsteady Bernoulli

$$\frac{p}{\rho} = \left[-\frac{\partial \phi_1}{\partial t} - \frac{\underset{\sim}{u_1}^2}{2} - gz - G_1(t)\right] = \left[-\frac{\partial \phi_2}{\partial t} - \frac{\underset{\sim}{u_2}^2}{2} - gz - G_2(t)\right] \quad \text{on } z=\eta(x,t) \quad *$$

Base state solution is

$$\underline{U} = \begin{cases} U_1 \underset{\sim}{i} & z>0 \\ U_2 \underset{\sim}{i} & z<0 \end{cases} \Rightarrow \begin{matrix} \phi_1 = U_1 x & z>0 \\ \phi_2 = U_2 x & z<0 \end{matrix}$$

$$P = \begin{cases} P_0 - \rho g z & z>0 \\ P_0 - \rho g z & z<0 \end{cases}$$

Introducing perturbations to base state:

$$\phi_1 = \phi_1' + U_1 x, \quad \phi_2 = \phi_2' + U_2 x, \quad p = P + p', \quad \eta = 0 + \eta' \quad \text{\& linearising}$$

Governing equation reduces to $\nabla^2 \phi_1' = 0 \quad z>0, \quad \nabla^2 \phi_2' = 0 \quad z<0$

Boundary conditions reduce to

$$\left.\begin{matrix} \dfrac{\partial \eta'}{\partial t} + U_1 \dfrac{\partial \eta'}{\partial x} = \dfrac{\partial \phi_1'}{\partial z} \\ \dfrac{\partial \eta'}{\partial t} + U_2 \dfrac{\partial \eta'}{\partial x} = \dfrac{\partial \phi_2'}{\partial z} \end{matrix}\right\} \quad \& \quad -U_1 \frac{\partial \phi_1'}{\partial x} - \frac{\partial \phi_1'}{\partial t} = -U_2 \frac{\partial \phi_2'}{\partial x} - \frac{\partial \phi_2'}{\partial t}$$

all applied on $z=0$.

Seek normal mode solutions of form $(\eta', \phi_{(2)}^{(1)}) = (\hat{\eta}, \hat{\phi}_{(2)}^{(1)}) e^{ikx+st}$

* [Moreover, require disturbance to be confined so that $\nabla\phi \to U\underset{\sim}{i}$ as $z \to \pm\infty$]

We obtain

$$\phi_1'(x) = Be^{-kz} e^{ikx+st} \quad \text{on ensuring } \nabla\phi' \to 0 \text{ as } z \to +\infty$$
$$\phi_2'(z) = Ce^{kz} e^{ikx+st} \quad \text{on ensuring } \nabla\phi' \to 0 \text{ as } z \to -\infty$$

Kinematic (linearised) b.c.'s give $C = (s + U_2 ik)\hat{\eta}/k$ & $B = -(s + U_1 ik)\hat{\eta}/k$

Dynamic (linearised) b.c. gives $\underline{2s^2 + 2ik(U_1+U_2) - k^2(U_2^2 + U_1^2) = 0}$

$$\Rightarrow \quad s = -\frac{ik}{2}(U_1+U_2) \pm 2k(U_1 - U_2).$$

**Q3: Assessor's comments**
The most popular question on the paper. Their attempts were very encouraging given this question required a linear stability analysis, a technique instrumental to the entire module.

4.

(a) A structure with a non-circular cross section experiences a fluid force that changes with the orientation of the flow. As a structure vibrates, its orientation changes and the fluid force oscillates. If this oscillatory force tends to increase vibration, the structure is unstable & large amplitude motion can result – this is galloping.

(b) Soft excitation

Under some conditions the mass will start to gallop from rest (from an angle of attack of zero) — this is soft excitation.

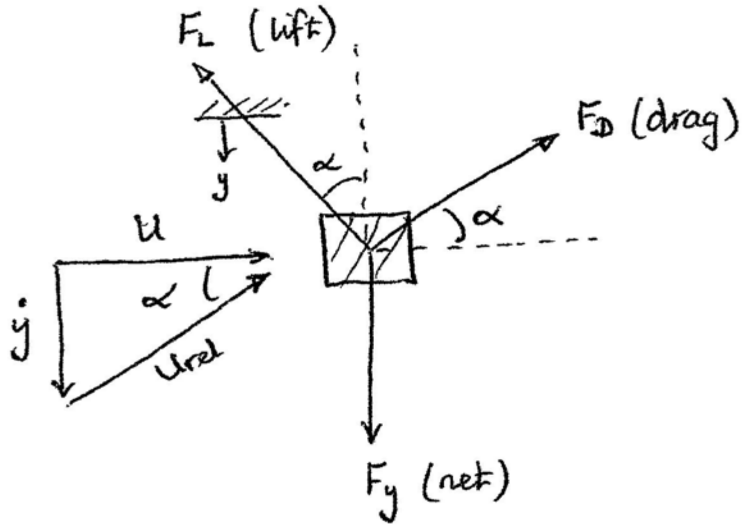
Hard excitation

If $\partial C_y/\partial \alpha > 0$ (a necessary, but not sufficient, condition for galloping instability) is met at some angle of attack, but not at zero angle of attack, galloping is still possible. However, an impulse is needed to excite the system & a critical flow velocity must be exceeded.

Hard excitation could be excited due to the motion caused by vortex shedding.

4

(c) . To show $C_y = -\dfrac{U_{rel}^2}{U^2}(C_L \cos\alpha + C_D \sin\alpha)$




$C_D$ is dimensionless drag force (drag coeff.) where

$$F_D = \tfrac{1}{2}\rho U_{rel}^2 D C_D \qquad \text{— (1)}$$

$C_L$ is dimensionless lift force (lift coeff.) where

$$F_L = \tfrac{1}{2}\rho U_{rel}^2 D C_L \qquad \text{— (2)}$$

Now as (from diagram), net aerodynamic force $F_y$ (positive downwards)

$$F_y = -F_L \cos\alpha - F_D \sin\alpha \qquad \text{— (3)}$$

We sub. for (1) & (2) to give

$$\underbrace{\tfrac{1}{2}\rho U^2 D C_y}_{F_y} = -\tfrac{1}{2}\rho U_{rel}^2 D C_L \cos\alpha - \tfrac{1}{2}\rho U_{rel}^2 D C_D \sin\alpha$$

$$\Rightarrow \quad C_y = -\frac{U_{rel}^2}{U^2}(C_L \cos\alpha - C_D \sin\alpha)$$

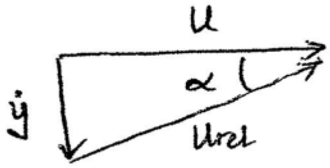
(d) A fluid force that is in phase with (ie. proportional to) the velocity can have the opposite effect of damping (the $\dot{y}$ term) — as energy is then pumped from the fluid to the structure. This may be regarded as a negative damping force.

4.

(d) cont$^{d}$: **To show $\partial C_y / \partial \alpha < 0$ for stability**

We need to express the RHS of the governing eq$^{n}$ ($m\ddot{y} + 2m\zeta\omega_y\dot{y} + ky = F_y$) as a damping term (i.e. in terms of $\dot{y}$), where RHS is $F_y = \frac{1}{2}\rho U^2 D C_y$ and $C_y = -\frac{U_{rel}^2}{U^2}(C_L\cos\alpha + C_D\sin\alpha)$.

Note: $\sin\alpha = \dfrac{\dot{y}}{U_{rel}} \simeq \alpha$ for small $\alpha$




$$
\begin{aligned}
U_{rel}^2 &= U^2 + \dot{y}^2 \\
&= U^2\left(1 + \frac{\dot{y}^2}{U^2}\right)^{1/2} = U^2\left(1 + \frac{1}{2}\frac{\dot{y}^2}{U^2} + \cdots\right), \quad |\dot{y}^2/U^2| < 1 \\
&\simeq U^2
\end{aligned}
$$

Hence $\alpha \simeq \dfrac{\dot{y}}{U_{rel}} \simeq \dfrac{\dot{y}}{U}$

We are concerned with small angles of attack $\alpha$, so lets examine net force coeff. expanded about a zero angle of attack,

Using a Taylor Series $\qquad C_y = C_y\big|_{\alpha=0} + \alpha \dfrac{\partial C_y}{\partial \alpha}\Big|_{\alpha=0} + \cdots$

Sub. this directly into

$$
\begin{aligned}
F_y &= \tfrac{1}{2}\rho U^2 D C_y \\
&= \tfrac{1}{2}\rho U^2 D\left[C_y\big|_{\alpha=0} + \underset{\dot{y}/U}{\underset{\uparrow}{\alpha}} \frac{\partial C_y}{\partial \alpha}\Big|_{\alpha=0} + \cdots\right]
\end{aligned}
$$

so to leading order

$$
F_y = \tfrac{1}{2}\rho U^2 D\, C_y\big|_{\alpha=0} + \underbrace{\tfrac{1}{2}\rho U D \frac{\partial C_y}{\partial \alpha}\Big|_{\alpha=0}\dot{y}}_{\text{proportional to velocity}}
$$

Sub. into governing eq$^{n}$. gives

$$
m\ddot{y} + 2m\zeta\omega_y\dot{y} + ky = \tfrac{1}{2}\rho U^2 D\, C_y\big|_{\alpha=0} + \tfrac{1}{2}\rho U D \frac{\partial C_y}{\partial \alpha}\Big|_{\alpha=0}\dot{y}
$$

$$
\Rightarrow \quad m\ddot{y} + 2m\omega_y \underbrace{\left(\zeta - \frac{\rho U D}{4m\omega_y}\frac{\partial C_y}{\partial \alpha}\Big|_{\alpha=0}\right)}_{\text{net damping term } \zeta_{net}}\dot{y} + ky = \tfrac{1}{2}\rho U^2 D\, C_y\big|_{\alpha=0}
$$

4

(d) cont$^{d}$. Net damping term is $\zeta_{net} = \left(\zeta - \frac{\rho U D}{4m\omega_y} \frac{\partial C_y}{\partial \alpha}\bigg|_{\alpha=0}\right)$ —(*)

i.e. governing eq$^{n}$ of form $m\ddot{y} + 2m\omega_y \zeta_{net} \dot{y} + ky = F$.

Solution of form $y = \frac{F(t=0)}{k} + e^{-\zeta_{net}\omega_y t} \sin[\quad]$ —(**)

$\Rightarrow$ Unstable if $\zeta_{net}$ negative.

Stable if $\zeta_{net}$ positive, which requires from (*)

that $\frac{\partial C_y}{\partial \alpha}\bigg|_{\alpha=0}$ and other $\alpha$ be negative (as this increases net stiffness)

Thus $\frac{\partial C_y}{\partial \alpha} < 0$ for stability

For instability we require $\zeta_{net}$ to be negative (see **)

Clearly $\frac{\partial C_y}{\partial \alpha} > 0$ to enable $\zeta_{net}$ to be negative (see *), however

For $\zeta_{net}$ to be negative, we require the velocity also to exceed a critical value given from (*) by

$$\zeta = \frac{\rho U_{crit} D}{4m\omega_y} \frac{\partial C_y}{\partial \alpha}\bigg|_{\alpha=0}$$

$$\Rightarrow U_{crit} = \frac{4m\omega_y \zeta}{\rho D} \Big/ \frac{\partial C_y}{\partial \alpha}\bigg|_{\alpha=0}$$

**Q4: Assessor's comments**

This question was also popular. It required the students to describe and analyse a galloping instability. Based on the average mark, the cohort understood well this aspect of the course.