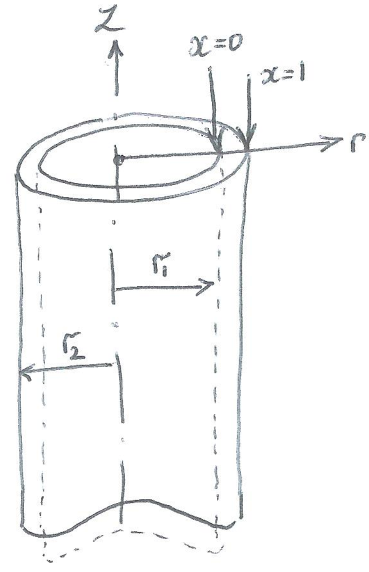
1

First, set $s=0$ for the marginal stability state &

(a) Seek solution to $\left[\frac{d^2}{dx^2}-(nd)^2\right]^3 \hat{u}_r = -T(nd)^2\hat{u}_r$ of the form $\hat{u}_r = \sin(N\pi x)$

where N is radial wavenumber.



Note that $\hat{u}_r = \sin(N\pi x)$ immediately satisfies the boundary conditions

$$\hat{u}_r = 0, \quad \frac{d^2\hat{u}_r}{dx^2}=0 \quad \& \quad \frac{d^4\hat{u}_r}{dx^4}=0 \quad \text{on } x=0,1.$$

& substituting into equation above gives, on noting that
$$\frac{d}{dx}\hat{u}_r = N\pi\cos(N\pi x) \quad \&$$
$$\frac{d^2}{dx^2}\hat{u}_r = -N^2\pi^2\sin(N\pi x) = -N^2\pi^2\hat{u}_r,$$
$$\left[-N^2\pi^2-(nd)^2\right]^3 \cancel{\hat{u}_r} = -T(nd)^2\cancel{\hat{u}_r}$$
$$\Rightarrow \quad T = \frac{1}{(nd)^2}\left[N^2\pi^2+(nd)^2\right]^3$$
is the condition for marginal stability

(b) We require the smallest value of $T=T_{crit}$ that gives rise to marginal stability. Given
$$T = \frac{1}{(nd)^2}\left[N^2\pi^2+(nd)^2\right]^3$$
the minimum value requires a radial wavenumber of $N=1$, so that
$$T = \frac{1}{(nd)^2}\left[\pi^2+(nd)^2\right]^3$$
The minimum required is that w.r.t the (dimensionless) vertical wavenumber (nd) [clear from question as given $u_r' = \hat{u}_r(r)\cos(nz).e^{st}$] with d the gap width. To find the minimum, we construct

$$\frac{dT}{d(nd)} = 0 \quad \Rightarrow \quad \frac{a^2.3[\pi^2+a^2]^2 2a - [\pi^2+a^2]^3 2a}{a^4} = 0$$
where $a=(nd)_{crit}$

So we require $\quad 6a^3(\pi^2+a^2)^2 = 2a(\pi^2+a^2)^3$

PTO.

1(b) contd

$$\Rightarrow \beta a^2 = \pi^2 + a^2$$
$$\therefore\ 2a^2 = \pi^2$$
$$\text{i.e. } a = (nd)_{crit} = \frac{\pi}{\sqrt{2}}$$

Thus, $T_{crit} = \frac{1}{(\pi/\sqrt{2})^2}\left[\pi^2 + \left(\frac{\pi}{\sqrt{2}}\right)^2\right]^3$

$$= \frac{2}{\pi^2}\,\frac{3^3\pi^6}{2^3}$$

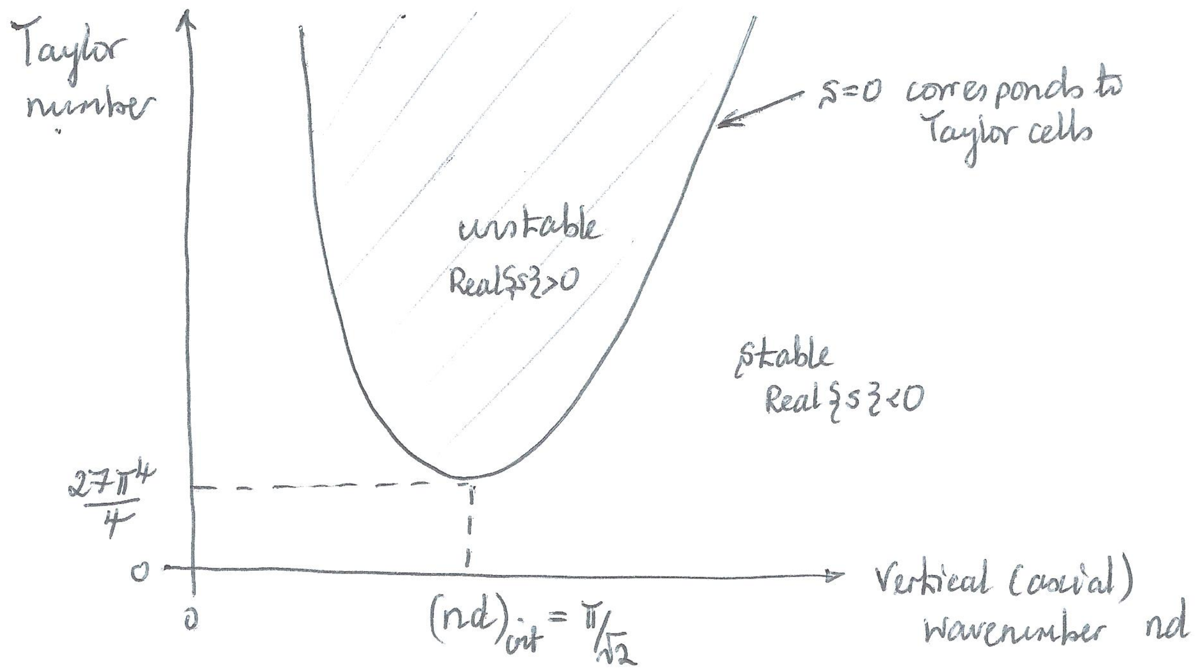
So, $T_{crit} = \frac{27}{4}\pi^4$ is the minimum value of the Taylor no. that gives rise to marginal stability.

Regarding the vertical structure of the flow. This critical Taylor no. would suggest structures with wavelength

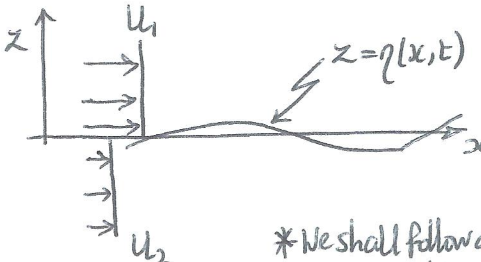
$$\lambda_{crit} = \frac{2\pi}{n_{crit}} = \frac{2\pi}{\left(\frac{\pi}{\sqrt{2}d}\right)} = 2\sqrt{2}d$$

So under given b.c's this would suggest vertical structure of wavelength $2\sqrt{2} \times$ gap width.

1(c).



2 continued II



*We shall follow a linear stability analysis

a). Given incompressible $\nabla\cdot\underset{\sim}{u}=0$. For fluid to support given velocity profile its inviscid, so $\underset{\sim}{u}=\nabla\phi$ for velocity potential ϕ ($\nabla\times\underset{\sim}{u}=0$) so

$$\nabla\cdot\underset{\sim}{u} = \nabla\cdot\nabla\phi = 0 \Rightarrow \underline{\nabla^2\phi=0}.$$

To assess the stability of the flow*, we introduce small amplitude perturbations to base flow & write

$$\begin{cases} \underset{\sim}{u} = \underset{\sim}{U} + u'(x,z,t) \\ p = P + p'(x,z,t) \\ \eta = 0 + \eta'(x,t) \end{cases} \quad \text{so that } \begin{cases} \phi_1 = U_1 x + \phi_1' \\ \phi_2 = U_2 x + \phi_2' \end{cases} \quad \text{as } u = \frac{\partial\phi}{\partial x}$$

where primed denote the perturbation quantities

Substituting into governing eq$^{\underline{n}}$ we have

$$\nabla^2(U_1 x+\phi_1') = 0+\nabla^2\phi_1' = 0$$

So that governing equations (they are already linear) that describe perturbation behaviour are

$$\begin{matrix} \nabla^2\phi_1'=0 & z>\eta'(x,t) \\ \nabla^2\phi_2'=0 & z<\eta'(x,t) \end{matrix} \quad \text{\& hence on linearising} \quad \underline{\begin{matrix} \nabla^2\phi_1'=0 & z>0 \\ \nabla^2\phi_2'=0 & z<0 \end{matrix}} \qquad (1)$$

Now consider the boundary conditions:

(a) In far field, far away from region of disturbance, we recover the base state, ie.

$$\begin{matrix} \nabla\phi_1 \to U_1 & \text{as } z\to\infty \\ \nabla\phi_2 \to U_2 & \text{as } z\to-\infty \end{matrix}$$

so that $\nabla(U_1x+\phi_1') \to U_1$ as $z\to\infty$

$\Rightarrow U_1 + \nabla\phi_1' \to U_1$

ie. $\nabla\phi_1' \to 0$ as $z\to\infty$

& similarly $\underline{\nabla\phi_2' \to 0 \text{ as } z\to-\infty}$ (2)

(b) Kinematic boundary condition (particles on interface, remain on interface). To develop this b.c. Define $F = z-\eta(x,t) = 0$

Thus $\frac{\partial F}{\partial t} + (\underset{\sim}{u}\cdot\nabla)F = \frac{DF}{Dt} = 0$, & with $u=\frac{\partial\phi}{\partial x}$, $w=\frac{\partial\phi}{\partial z}$

We have $\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial z}$, $\phi=\{\phi_1,\phi_2\}$ on $z=\eta(x,t)$.

On introducing the perturbation quantities, we obtain

e.g. $\frac{\partial\eta'}{\partial t} + \frac{\partial(U_1x+\phi_1')}{\partial x}\frac{\partial\eta'}{\partial x} = \frac{\partial(U_1x+\phi_1')}{\partial z}$ on $z=\eta'(x,t)$

& so neglecting products of small terms, evidently

$$\underline{\frac{\partial\eta'}{\partial t} + U_1\frac{\partial\eta'}{\partial x} = \frac{\partial\phi_1'}{\partial z} \quad \& \quad \frac{\partial\eta'}{\partial t} + U_2\frac{\partial\eta'}{\partial x} = \frac{\partial\phi_2'}{\partial z} \quad \text{on } z=0} \qquad (3).$$

2. (a) cont^{d}.F

Final boundary condition is the dynamic b.c. (pressure continuous across interface). For unsteady irrotational flow

$$\frac{\partial \phi_1}{\partial t} + \frac{p}{\rho_1} + \frac{1}{2}\underset{\sim}{u}^2 + gz = \hat{G}_1(t), \text{ so for pressure to be continuous across interface}$$

We can write

$$p = -\rho_1\left[\frac{\partial \phi_1}{\partial t} + \frac{\underset{\sim}{u}^2}{2} + gz + G_1\right] = -\rho_2\left[\frac{\partial \phi_2}{\partial t} + \frac{\underset{\sim}{u}^2}{2} + gz + G_2\right] \quad \text{for some } G_1 \,\&\, G_2.$$

In the base state, the interface is at $z=0$, the flow is steady so $\partial/\partial t \equiv 0$, & we have

$$\rho_1\left(\frac{U_1^2}{2} + G_1\right) = \rho_2\left(\frac{U_2^2}{2} + G_2\right). \qquad \text{— } (*)$$

Sub. for perturbation quantities

$$\rho_1\left[\frac{\partial}{\partial t}(U_1x+\phi_1') + \frac{(\nabla\phi_1)^2}{2} + g\eta' + G_1\right] = \rho_2\left[\frac{\partial}{\partial t}(U_2x+\phi_2') + \frac{(\nabla\phi_2)^2}{2} + g\eta' + G_2\right]$$

on $z=\eta'(x,t)$

Now $\frac{(\nabla\phi_1)^2}{2} = \frac{1}{2}[\nabla(U_1x+\phi_1')]^2 = \frac{1}{2}(U_1+\nabla\phi_1')^2 = \frac{1}{2}\left[U_1^2 + 2U_1\frac{\partial\phi_1'}{\partial x} + (\nabla\phi_1')^2\right]$

$$= \tfrac{1}{2}\left[U_1^2 + 2U_1\frac{\partial\phi_1'}{\partial x}\right] \text{ on linearising}$$

Thus,

$$\rho_1\left[\frac{\partial\phi_1'}{\partial t} + \frac{U_1^2}{2} + U_1\frac{\partial\phi_1'}{\partial x} + g\eta' + G_1\right] = \rho_2\left[\frac{\partial\phi_2'}{\partial t} + \frac{U_2^2}{2} + U_2\frac{\partial\phi_2'}{\partial x} + g\eta' + G_2\right]$$

on $z=0$ (linearising)

& Using $(*)$

$$\underline{\rho_1\left[\frac{\partial\phi_1'}{\partial t} + U_1\frac{\partial\phi_1'}{\partial x} + g\eta'\right] = \rho_2\left[\frac{\partial\phi_2'}{\partial t} + U_2\frac{\partial\phi_2'}{\partial x} + g\eta'\right] \text{ on } z=0.}$$

We now seek normal mode solutions of form

$$\eta'(x,t) = \hat{\eta}\, e^{ikx+st}$$
$$\phi_1'(x,z,t) = \hat{\phi}_1(z)\, e^{ikx+st} \quad z>0$$
$$\phi_2'(x,z,t) = \hat{\phi}_2(z)\, e^{ikx+st} \quad z<0 \quad \& \text{ sub. into } \nabla^2\phi'=0$$

to give

$$\frac{d^2\hat{\phi}_1}{dz^2} - k^2\hat{\phi}_1 = 0 \Rightarrow \hat{\phi}_1 = Ae^{kz} + Be^{-kz}.$$

Bounded disturbance b.c. gives

$$\underline{\hat{\phi}_1 = Be^{-kz} \text{ and similarly } \hat{\phi}_2 = Ce^{kz}}$$

Thus $\phi_1' = Be^{-kz}e^{ikx+st}$ & $\phi_2' = Ce^{kz}e^{ikx+st}$.

Using kinematic b.c.'s

$$s\hat{\eta}\cancel{e^{ikx+st}} + U_1 ik\hat{\eta}\cancel{e^{ikx+st}} = -kBe^{-kz}\cancel{e^{ikx+st}} \quad \text{on } z=0$$
$$\Rightarrow B = -\frac{\hat{\eta}}{k}(s+ikU_1).$$
$$s\hat{\eta}\cancel{e^{ikx+st}} + U_2 ik\hat{\eta}\cancel{e^{ikx+st}} = Cke^{kz}\cancel{e^{ikx+st}} \quad \text{on } z=0.$$
$$\Rightarrow C = \hat{\eta}/k\,(s+ikU_2).$$

contd III.

2 (a) Finally using dynamic b.c. $\left\{ \begin{array}{l} \text{recall } \phi_1' = -\hat{\eta}/_k (s+ikU_1)e^{-kx} e^{ikx+st} \\ \phi_2' = \hat{\eta}/_k (s+ikU_2)e^{kx} e^{ikx+st} \end{array}\right.$

$$\rho_1\left[-\frac{\hat{\eta}}{k}(s+ikU_1)\cancel{e^{-kx}} s \cancel{e^{ikx+st}} + U_1\left(-\frac{\hat{\eta}}{k}\right)(s+ikU_1)\cancel{e^{-kx}} ik \cancel{e^{ikx+st}} + \hat{\eta} g \cancel{e^{ikx+st}}\right]$$

$$= \rho_2\left[\frac{\hat{\eta}}{k}(s+ikU_2)\cancel{e^{kx}} s \cancel{e^{ikx+st}} + U_2 \frac{\hat{\eta}}{k}(s+ikU_2)\cancel{e^{kx}} ik \cancel{e^{ikx+st}} + \hat{\eta} g \cancel{e^{ikx+st}}\right] \text{ on } z=0$$

$\times\left(\frac{k}{\hat{\eta}}\right)$ $\Rightarrow$ $\underline{\rho_1\left(kg-(s+iU_1k)^2\right) = \rho_2\left(kg+(s+iU_2k)^2\right)}$

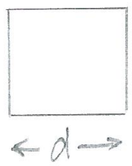
(b) Solving this quadratic gives growth rate

$$s = \underbrace{-ik\,\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1+\rho_2}} \pm \underbrace{\left[k^2\,\frac{\rho_1\rho_2(U_1-U_2)^2}{(\rho_1+\rho_2)^2} - gk\,\frac{(\rho_2-\rho_1)}{\rho_1+\rho_2}\right]^{1/2}}_{\text{real part.}}$$

Real $\{s\} > 0$ thus requires $\quad k > \dfrac{g\,(\rho_2^2-\rho_1^2)/\rho_1\rho_2}{(U_1-U_2)^2}$

$\Rightarrow$ larger density differences stabilise low k disturbances, greater shear increases range of k that result in instability.

3



$d = 0.24\,m$

$m_a = 70\ kg\,m^{-1}$

$\omega_a = 315\ rad\,s^{-1}$ when not submerged

$\zeta = 0.01$ when submerged

↑ defined s.t. $m\ddot{y} + 2m\zeta\omega\dot{y} + ky = 0$

$\rho_w = 1025\ kg\,m^{-3}$

added mass per unit length $= 1.51\,\rho_w \pi d^2$

$c_y = 2\alpha$ for this shape ; $\Rightarrow \dfrac{\partial c_y}{\partial \alpha} = 2$

a) submerged strut. We require the added mass to get the new mass per unit length, m_w: $m_w = m_a + 1.51\,\rho_w \pi d^2$

$$= 70 + 1.51 \times 1025 \times \pi \times 0.24^2$$

$$= 350\ kg\,m^{-3}$$

The fluid force on the structure is $\frac{1}{2}\rho U^2 \frac{\partial c_y}{\partial \alpha}\alpha$ where $\alpha \simeq \frac{\dot{y}}{U}$

The equation of motion is: $m_w\ddot{y} + 2m_w\zeta\omega_w\dot{y} + ky = \frac{1}{2}\rho U^2 \frac{\partial c_y}{\partial \alpha}\frac{\dot{y}}{U}d$

This will start to gallop from soft excitation when:

$$\frac{1}{2}\rho U \frac{\partial c_y}{\partial \alpha} d > 2m_w\zeta\omega_w$$

The stiffness doesn't change when submerged, so $\dfrac{\omega_w}{\omega_a} = \left(\dfrac{m_a}{m_w}\right)^{1/2}$

$$\Rightarrow \quad U > \frac{4 m_w \zeta \left(\frac{m_a}{m_w}\right)^{1/2} \omega_a}{\rho \, \partial c_y/\partial\alpha \, d} = \frac{4 \times 350 \times 0.01 \times \left(1/5\right)^{1/2} \times 315}{1025 \times 2 \times 0.24}$$

$$= 4.01\ m/s$$

The fluid causes a force on the body in the direction of its motion. This force is, for small amplitudes, proportional to α, the apparent angle of attack. In turn this is proportional to $\dot{y}/U$. This force acts as "negative damping". When this exceeds the mechanical damping, oscillations will start from rest.

b) The mass/spring/damper system will oscillate, driven by the strut's oscillation. The amplitude of oscillation will be greatest when the natural frequency of the mass/spring/damper system is closest to the natural frequency of the strut. This means that the damping will also be greatest ~~then~~ when the two frequencies are the same. Provided that the mass/spring/damper system is not itself driven by the flow, it will remove energy at the strut's natural frequency, thereby reducing the oscillation amplitude of the strut.

The target frequency is $\omega_w = \left(\frac{m_a}{m_w}\right)^{1/2} \times \omega_a = \left(\frac{1}{5}\right)^{1/2} \times 315 = 140.9$ rad/sec

The mass/spring system has resonant frequency $\simeq \sqrt{\frac{k}{m}}$

$$\Rightarrow \cancel{m \simeq k\,(140.9)^2} \Rightarrow k/(140.9)^2 = m = 1.01\,\text{kg}$$

Check units: $\frac{N m^{-1}}{s^{-2}} = N m^{-1} s^{+2} = (kg\, m s^{-2}) m^{-1} s^{+2} = kg$ ✓ OK.

c) The mass/spring/damper system would be sensible if the strut needed to be electrically insulated from the surroundings (which is why these are used on electricity power lines), but it is too complicated for this system. Here, the strut could simply be attached to the other struts with rubber dampers, or stiffened. Alternatively it could be streamlined to avoid gallop (although this may cause flutter) or replaced with a strut with circular cross-section (although this may cause vortex shedding).

4. a) Large amplitude varicose oscillations are visible in the image. Similarly, a strong spectral peak at 1000 Hz and its overtone at 2000 Hz is visible in the P.S.D. This shows that the jet is oscillating at a well-defined frequency, in a varicose manner.

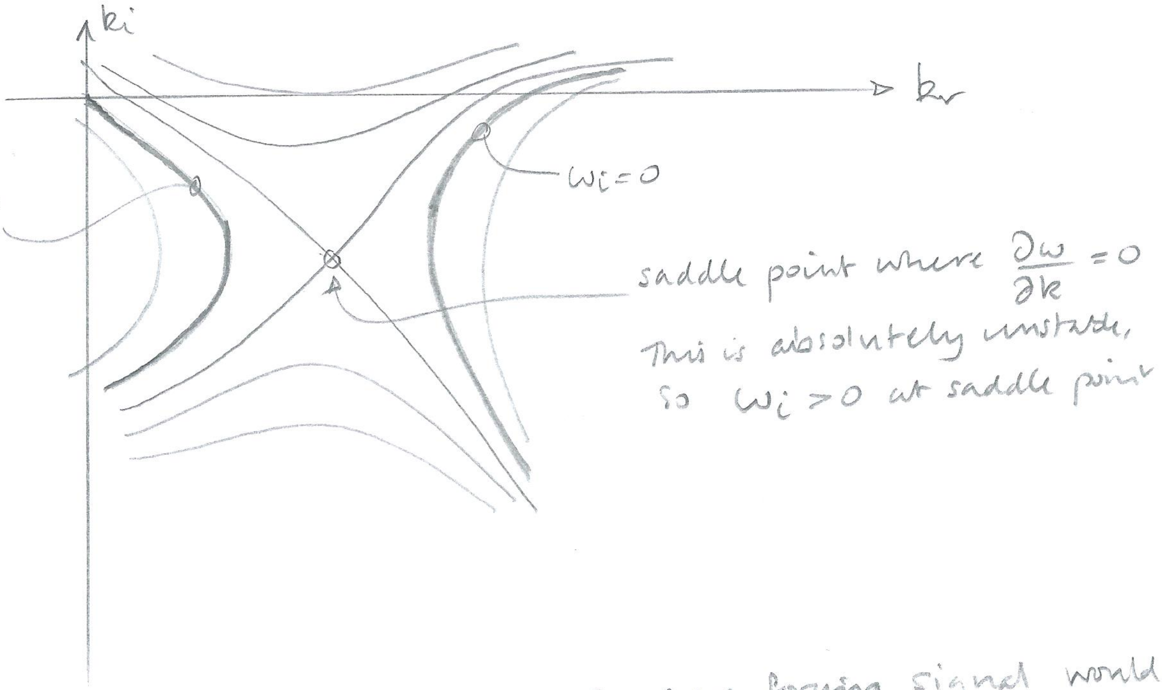
b) This oscillation arises because the flow is absolutely unstable over a sufficiently large region to make the corresponding steady flow ~~absolutely~~ globally unstable. This global instability causes the oscillations observed to grow to the limit cycle that is observed. (Downstream, it is likely that the flow is convectively unstable, but this is less relevant.)

c) k_r is the axial wavenumber: $2\pi/\lambda$ where λ is the wavelength

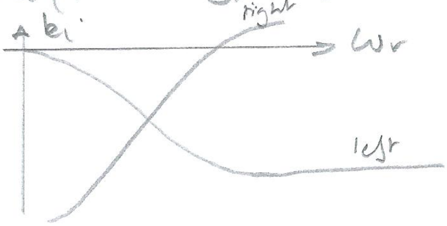
k_i is the axial decay rate of spatial oscillations

ω_r is the angular frequency of oscillations

ω_i is the temporal growth rate of oscillations



d) The flow is absolutely unstable here, so any forcing signal would be drowned out by the impulse response at that point (in the linear analysis) - ie. by the oscillating global mode (in reality). If one attempts a spatial analysis, one obtains: for the left branch & right branch.



ASSESOR's COMMENTS, MODULE 4A10

Question 1
This question was centred around the stability of a Taylor-Couette flow and required the students to recognise that they needed to set the growth rate to zero in order to establish the marginal stability condition that the question demanded. Although this question required some thought before launching into it, to my surprise just less than half the cohort attempted it.

Question 2
Circa one half of the module is dedicated to being able to perform a classic linear stability analysis and this question required the students to apply such an analysis to a two-layer density stratified system with shear. All attempted this question, overall doing very well with an average of 14/20. The analysis is not straightforward and the students showed their command of the subject with almost all making solid and complete attempts at the question.

Question 3
This was a conceptual and numerical question about gallop and added mass.

(a) This was reasonably well answered by most candidates. Almost all candidates correctly identified gallop as the instability mechanism but many of these did not convincingly explain the physical mechanism. This is not difficult, taking perhaps 4 sentences, but it requires clarity of thought and argument. Most candidates correctly included added mass but many forgot to account for the shift in resonant frequency when the strut is submerged.

(b) Most candidates correctly identified that the extra mass acts as a tuned mass damper, although some did not explain the mechanism well: the extra damping arises due to the damping in the tuned mass damper, not because it has extra mass.

(c) This section was well answered by almost all candidates, showing good physical understanding of how to prevent fluid-structure oscillations of this type.

Question 4
This was a conceptual and descriptive question about absolute/convective instability in a hot jet. It did not require any calculations. Many students answered it well, although several did not read (a) & (b) carefully enough.

(a) & (b) This was very well answered by around 20% of students, who described the motion of the jet accurately and explained how it arises using the concepts of absolute and convective instability. Around 50% of the students did not answer the question. Instead, they quoted the definitions of absolute and convective instability without applying them to the hot jet. This was an example of why it is important to read the question being asked, not the question that one thinks is being asked.

(c) Almost every student correctly identified the physical meaning of real(k), imag(k), real(omega), and imag(omega). The handful that got this wrong did so by confusing k and omega, and real and imaginary. Well over 50% of the students correctly drew the sketch of imag(omega) contours in the complex k-plane, which was the main part of the question. There were some excellent answers to this question.

(d) Around 75% of students correctly answered this question, showing good understanding of the concepts.