1 (a)

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u}, \qquad \frac{D\mathbf{u}}{Dt} = -\nabla\left(\frac{p}{\rho}\right)$$

$$\Rightarrow \quad \frac{D}{Dt}(\mathbf{u}\cdot\boldsymbol{\omega}) = \mathbf{u}\cdot\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega}\cdot\frac{D\mathbf{u}}{Dt} = \mathbf{u}\cdot\left[\boldsymbol{\omega}\cdot\nabla\mathbf{u}\right] + \boldsymbol{\omega}\cdot\left[-\nabla\left(\frac{p}{\rho}\right)\right]$$

$$= \boldsymbol{\omega}\cdot\nabla\left(\tfrac{1}{2}u^2\right) - \nabla\cdot\left(\frac{p}{\rho}\boldsymbol{\omega}\right)$$

$$= \nabla\cdot\left[\left(\tfrac{1}{2}u^2 - \frac{p}{\rho}\right)\boldsymbol{\omega}\right]$$

(noting $\nabla\cdot\boldsymbol{\omega} = 0$)

$$\frac{d}{dt}\int(\mathbf{u}\cdot\boldsymbol{\omega})\,dV = \int\frac{D}{Dt}(\mathbf{u}\cdot\boldsymbol{\omega})\,dV + \int(\mathbf{u}\cdot\boldsymbol{\omega})\frac{D(dV)}{Dt}$$

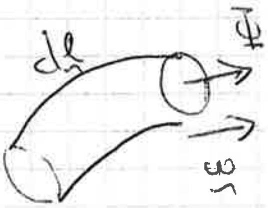
If dV is a material volume, then $D(dV)/Dt = 0$.

$$\Rightarrow \quad \frac{d}{dt}\int\mathbf{u}\cdot\boldsymbol{\omega}\,dV = \int\nabla\cdot\left[\left(\tfrac{1}{2}u^2 - \frac{p}{\rho}\right)\boldsymbol{\omega}\right]dV$$

$$= \oint_S\left[\left(\tfrac{1}{2}u^2 - \frac{p}{\rho}\right)\boldsymbol{\omega}\right]\cdot d\mathbf{S} \quad \text{(using Gauss)}$$

If $\boldsymbol{\omega}$ is localised, $\boldsymbol{\omega} = 0$ on S, so the surface integral is ze

$$\Rightarrow \quad \frac{d}{dt}\int\mathbf{u}\cdot\boldsymbol{\omega}\,dV = 0.$$



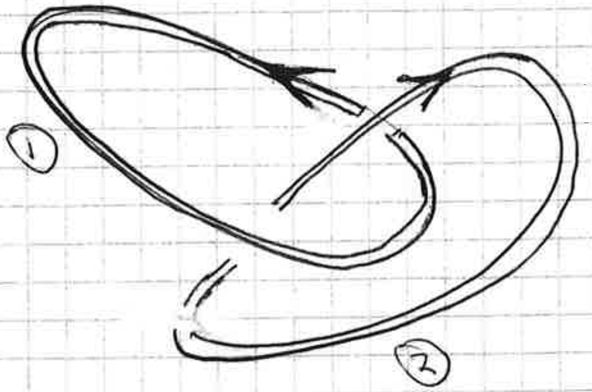
(b) (i) $$H = \int\mathbf{u}\cdot\boldsymbol{\omega}\,dV = \int_{V_1}\mathbf{u}\cdot\boldsymbol{\omega}\,dV + \int_{V_2}\mathbf{u}\cdot\boldsymbol{\omega}\,dV$$

$$= \oint_{C_1}\mathbf{u}\cdot(\Phi_1\,d\mathbf{l}) + \oint_{V_2}\mathbf{u}\cdot(\Phi_2\,d\mathbf{l})$$

(since $\boldsymbol{\omega}\,dV = \boldsymbol{\omega}A|d\mathbf{l}| = \Phi\,d\mathbf{l}$)

But Φ is constant along a vortex tube and so,

$$H = \Phi_1\oint_{C_1}\mathbf{u}\cdot d\mathbf{l} + \Phi_2\oint_{C_2}\mathbf{u}\cdot d\mathbf{l}$$



From Stokes,

$$\oint_{c_1} \underset{\sim}{u}\cdot d\underset{\sim}{l} = \Phi_2 \,,\quad \oint_{c_2} \underset{\sim}{u}\cdot d\underset{\sim}{l} = \Phi_1$$

(if Right-Handed linkage)

Thus $H = \Phi_1\Phi_2 + \Phi_2\Phi_1 = 2\Phi_1\Phi_2$

(ii) If the linkage is left-handed, then

$$\oint_{c_1} \underset{\sim}{u}\cdot d\underset{\sim}{l} = -\Phi_2 \,,\quad \oint_{c_2} \underset{\sim}{u}\cdot d\underset{\sim}{l} = -\Phi_1 \Rightarrow \underline{\underline{H = -2\Phi_1\Phi_2}}$$

If they are not linked, $\oint_{c_1} \underset{\sim}{u}\cdot d\underset{\sim}{l} = \oint_{c_2} \underset{\sim}{u}\cdot d\underset{\sim}{l} = 0 \Rightarrow H = 0$

(c) Helmholtz's laws:

(I) The vortex lines are frozen into the fluid (ie fluid elements on a vortex line stay on that line)

(ii) The flux of vorticity along a vortex tube is independent of time and constant along the tube.

In (b) the flux in tubes 1 and 2 is conserved (H I) and the vortex tubes are frozen into the fluid (H II). Since the vortex tubes are frozen into the fluid they cannot change their topology (ie. their linkage). Thus $H = +2\Phi_1\Phi_2, -2\Phi_1\Phi_2, 0$ for all time.

(d) If the fluid is viscous the vortex tubes are <u>not</u> frozen into the fluid, and may diffuse through the fluid. Thus the linkage of the two tubes need not be conserved for all time, and hence H need not be conserved.

2(a) : $\underset{\sim}{u} \cdot \nabla T = \alpha \nabla^2 T$

(i) If $\alpha = 0$ $\quad \underset{\sim}{u} \cdot \nabla T = 0 \Rightarrow \nabla T$ is $\perp^{\text{ar}}$ to $\underset{\sim}{u}$

But ∇T is $\perp^{\text{ar}}$ to isotherms

$\Rightarrow$ Isotherms $\parallel^{\text{le}}$ to streamlines

$\Rightarrow$ $\underline{\underline{T = T(\psi)}}$.

$$T = T(\psi) \Rightarrow \nabla T = T'(\psi) \nabla\psi$$

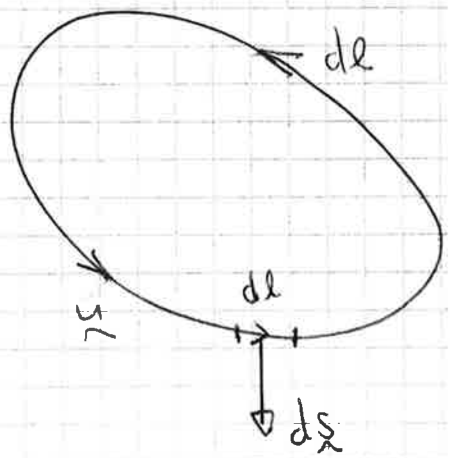
(Chain rule)

$$\Rightarrow \quad \underset{\sim}{u} \cdot \nabla T = \alpha \nabla \cdot (\nabla T) = \underline{\underline{\alpha \nabla \cdot (T'(\psi) \nabla \psi)}}$$

(ii)

$$\int \underset{\sim}{u} \cdot \nabla T \, dA = \alpha \int \nabla \cdot (T'(\psi) \nabla \psi) \, dA$$

$$\Rightarrow \int \nabla \cdot [T \underset{\sim}{u}] \, dA = \alpha \int \nabla \cdot [T'(\psi) \nabla \psi] \, dA$$



Apply Gauss' theorem

$$\oint_C T \underset{\sim}{u} \cdot d\underset{\sim}{s} = \alpha \oint_C T'(\psi) \nabla \psi \cdot d\underset{\sim}{s}$$

where $\quad d\underset{\sim}{s} = \underset{\sim}{n} \, dl$

$$\Rightarrow \quad \oint_C T \underset{\sim}{u} \cdot \underset{\sim}{n} \, dl = \alpha \oint_C T'(\psi) \nabla \psi \cdot \underset{\sim}{n} \, dl$$

But $\underset{\sim}{u} \cdot \underset{\sim}{n} = 0$ as C is a streamline.

Also, ψ is constant on C, so $T'(\psi)$ is constant on C

Therefore $\quad \underline{\underline{0 = \alpha T'(\psi) \oint_C \nabla \psi \cdot \underset{\sim}{n} \, dl}}$

(iii) $$\oint_C \nabla\psi \cdot \underset{\sim}{n}\, dl = \oint_C |\nabla\psi|\, dl \neq 0$$

Thus $$\underbrace{\alpha}_{\neq 0}\, T'(\psi) \underbrace{\oint_C |\nabla\psi|\, dl}_{\neq 0} = 0$$

As α and $\oint |\nabla\psi| dl$ are non zero, we have $T'(\psi) = 0$

Thus T is constant on a streamline and does not vary across a streamline. So $T = \text{const.}$, for small but finite α.

(b) The physical interpretation is that $\underset{\sim}{u} \cdot \nabla T \approx 0$ tells us T is approximately constant along streamlines (because of convection). However, we have a small but finite diffusion of vorticity that slowly erradicates gradients between streamlines, so the final steady state is $T = \text{const}$.

For steady-on-average turbulent flow turbulent diffusion takes the place of laminar diffusion, so α is replaced by α_T. The end result is the same.

The Batchelor - Prandtl Theorem says that for 2D flows at large Re, which are laminar, have $\omega = \text{const.}$ outside boundary layers, where $\underset{\sim}{\omega} = \nabla \wedge \underset{\sim}{u}$. The reason is that ω is governed by $$(\underset{\sim}{u} \cdot \nabla)\omega = \nu \nabla^2 \omega$$
which is identical to the temperature equation, but with ν replacing α and ω replacing T. Also $Re \gg 1 \Rightarrow$ small ν. Thus we may repeat the analysis of (a) to give $\omega = \text{const}$.

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3 (a) $\frac{\partial \sigma^2}{\partial t} + \bar{u}_j \frac{\partial \sigma^2}{\partial x_j} = \frac{\partial}{\partial x_i}\left[(D + D_{turb}) \frac{\partial \sigma^2}{\partial x_i}\right] + 2 D_{turb} \left(\frac{\partial \bar{\Phi}}{\partial x_i}\right)^2 - 2\bar{N}$

① ② ③ ④ ⑤ ⑥

Model for $\bar{N} = C \frac{\sigma^2}{L_t / u}$

L_t : integral lengthscale

u : typical velocity fluctuation ($u \approx \sqrt{k}$, k: turb kinetic energy)

C : constant $o(1)$

Unsteady term ① : $\frac{\Sigma^2}{T}$ T depends on time varying mean flow

② : Advection by mean flow : $U \frac{\Sigma^2}{L_t}$

③ : Molecular diffusion : $D \frac{\Sigma^2}{L_t^2}$ D : diffusivity

④ : turbulent diffusion : $u L_t \frac{\Sigma^2}{L_t^2}$ $D_{turb} \sim o(u L_t)$

⑤ : production : $u L_t \frac{\Phi^2}{L_t^2}$

⑥ : Scalar dissipation : $\Sigma^2 \frac{u}{L_t}$

Φ : order of magnitude of mean scalar

Σ : order of magnitude of rms scalar

Physical origin of $\bar{N}$ model is energy cascade : (scalar energy)/(eddy turnover time) = scalar dissipation

3(b). For the simplified problem given

$$\frac{\partial \sigma^2}{\partial t} = -2C \frac{\sigma^2}{T_{turb}} \qquad T_{turb} = \frac{L_{turb}}{u}$$

$$\Leftrightarrow \quad \sigma^2 = \sigma^2(t=0) \exp\left[-\frac{2C}{T_{turb}} t \right]$$

If $\frac{\sigma^2}{\sigma^2(t=0)} = 0.1$, $\Rightarrow$ $\boxed{t = 2.3 \frac{T_{turb}}{2C}}$

If $C = 1$, $\Rightarrow \frac{t}{T_{turb}} = 1.15$

(c): If statistically steady: k does not depend on time.

If all quantities are homogeneous & turbulence is isotropic, there is no production mechanism (e.g shear, buoyancy), so nothing balances ε (dissipation).

So either k must be decaying, or there must be a production mechanism which invariably leads to inhomogeinity and/or anisotropy.

4 (a) θ' produces $\overline{u_3'^2}$, i.e. the vertical component because there is no production of buoyancy for the horizontal components. This is because the fluctuating body force $\overline{g_i u_i'}$ exists only in the vertical direction.

The other components receive their energy from the pressure-strain correlation term the Re-stress equation.

(b) If there is no shear & no turbulent & molecular transport of k, then the k equation becomes

$$\begin{pmatrix}\text{Production by} \\ \text{buoyancy}\end{pmatrix} = (\text{dissipation})$$

$$\Rightarrow \quad \frac{g}{T}\overline{\theta' u'} = \varepsilon$$

The order of magnitude of $\overline{\theta' u'}$ is Θu where Θ is order of magnitude of the temperature fluctuations.

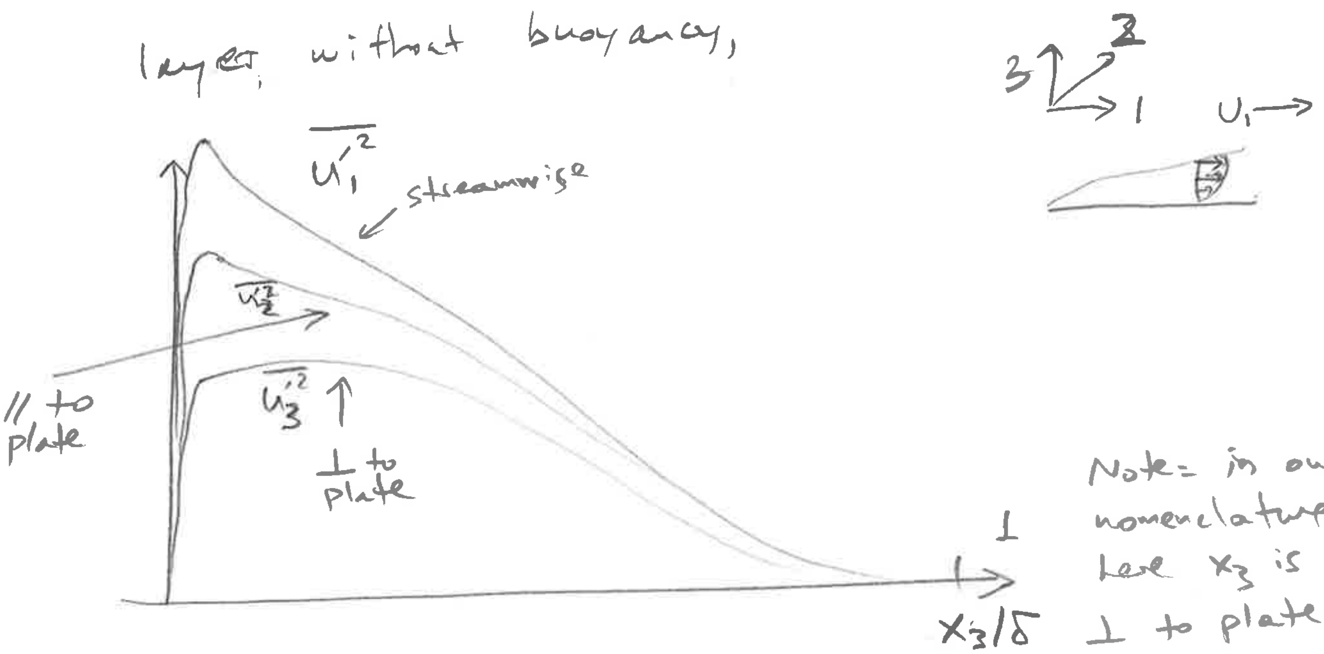
$$\Rightarrow \quad \frac{g}{T}\Theta u \sim \frac{u^3}{L}$$

L: integral lengthscale

$k = u^2$

$$\Rightarrow \quad \boxed{\Theta = O\left(\frac{kT}{gL}\right)}$$

(c) For a flat plate turbulent boundary layer, without buoyancy,



Note: in our nomenclature here x_3 is ⊥ to plate

At the inner layer, all components → 0 at the wall

If mean Θ follows mean u profile, there is production of θ'; hence production of u_3'. So above picture changes, with $\overline{u_3'^2}$ curve to probably be closer to the other two. Depending on the strength of the buoyant relative to the mechanical production, the anisotropy could be small or large. In general, the $\overline{\theta'^2}$ will follow the vertical component, as it is the <u>vertical</u> turbulent flux $\overline{u_3'\theta'}$ that multiplies $\frac{\partial \overline{\Theta}}{\partial x_3}$ to produce the $\overline{\theta'^2}$.

Examiner's comments

Q1 Helicity and vortex tubes
A popular and straightforward question, well-answered by most candidates. Not everyone explained the physical interpretation in (d) correctly.

Q2 Temperature field in 2-D
A popular and straightforward question, well-answered by most candidates.

Q3 Scalar fluctuations
Generally satisfactory performance, but few candidates described the physics of the problem accurately. In (b), many candidates confused the scalar dissipation with dissipation of kinetic energy resulting in a factor of 2 error.

Q4 Hot plate boundary layer
An original question. Part (b) was very well done, the physical description of the phenomena was not refined enough.