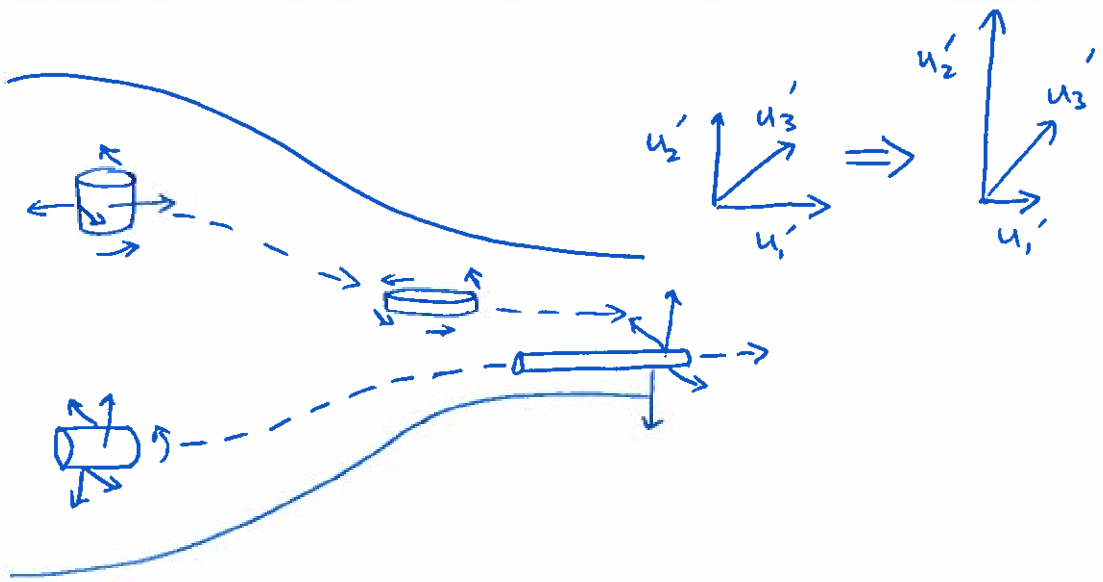
Question 1.

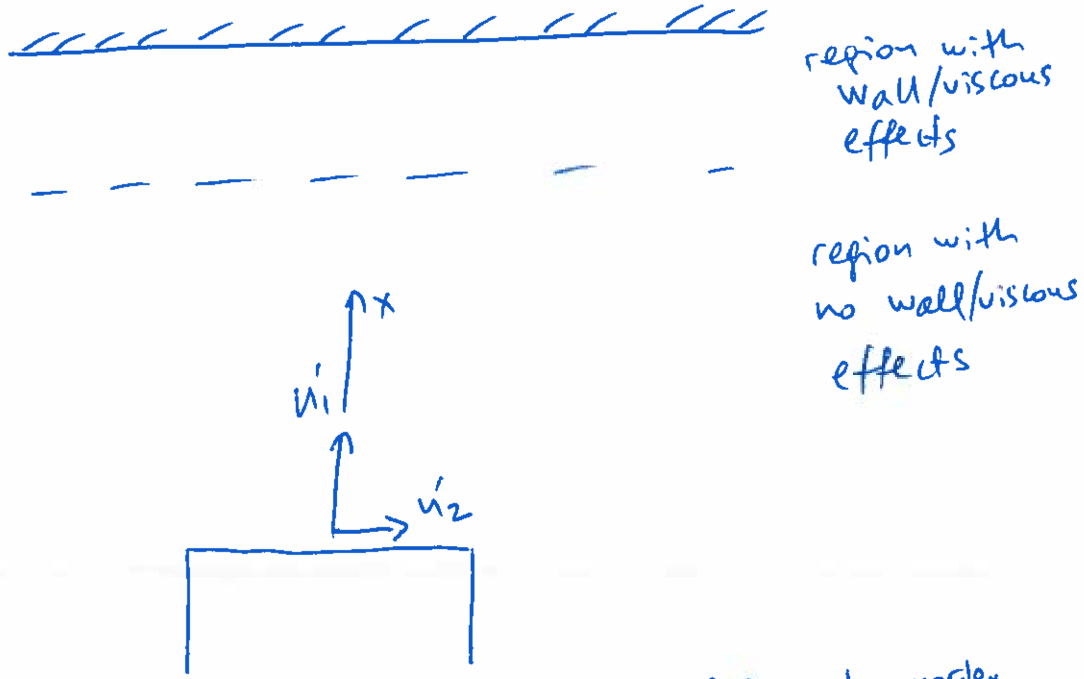
(a) Vortex stretching is a three-dimensional, essentially inviscid, mechanism by which the flow stretches a vortex line, hence compressing its width and extending its length, while increasing its angular velocity. The natural instability of a vortex line makes a collection of such lines entangled and three-dimensional. The result is to concentrate the vorticity in a small fraction of the whole flow volume. The smallest scales of the turbulent flow, the Kolmogorov scales, can be thought of crudely as thin vortex lines containing most of the vorticity of the flow. The vorticity amplification due to vortex stretching is eventually balanced by viscous effects at the Kolmogorov scales.

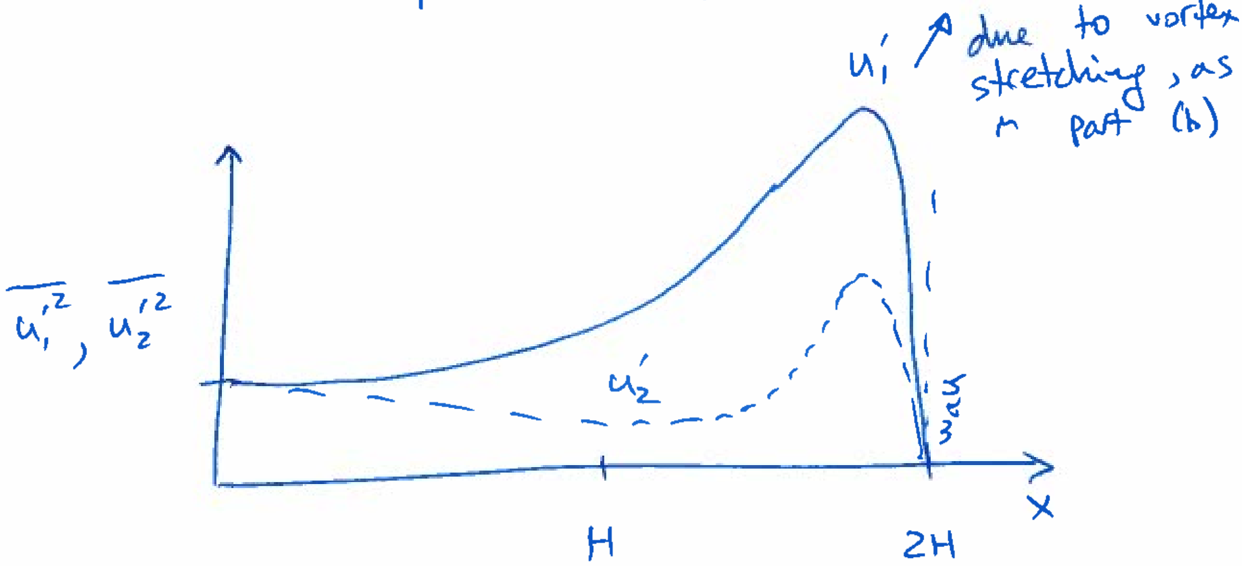
(b) The flow is as sketched below, with eddy shapes approximately marked by the disks shown and the individual components of velocity magnified or reduced according to the length of the arrows:



Being essentially an inviscid process, eddy distortion due to the mean streamline pattern is as shown in the diagram. The volume of the barrel-shaped eddy at the entry of the nozzle is conserved, and the angular momentum is also conserved, which means the individual velocity components are affected as shown above. It is not very clear how u_3' will behave from this simple argument. But considering that an eddy orientated as in the upper sketch would increase u_3' while an eddy oriented as in the lower one would decrease u_3', the experimental observation that u_3' is not affected too much by the contraction can be explained.

(c)





u_2' initially decreases, but eventually close to the wall it increases as in a boundary layer. Eventually, both components $\to 0$ at the wall.

Question 2.

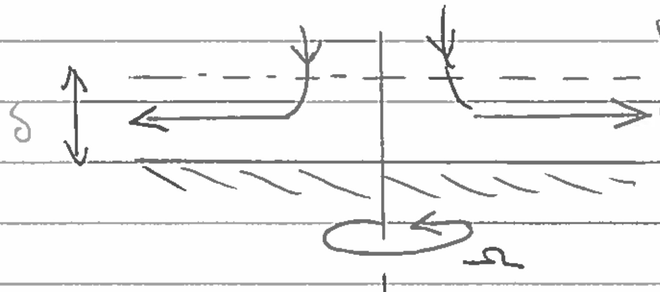
(a) $P_0 = \varepsilon$ for statistically steady turbulence, and since in general $\varepsilon = k^{3/2}/L$, $L_0 = k_0^{3/2}/P_0$.

(b) The idealisation of homogeneous isotropic non-decaying turbulence cannot be realised in practice. First, there are wall effects which will create thin unsteady boundary layers and local turbulence anisotropy and inhomogeneities. Second, close to the fans the flow will have a mean component associated with the fan motion and there will be shear-induced turbulence. Finally, if everything is fully homogeneous, there cannot be energy input to the turbulence to balance the dissipation (in the absence of a body force). So the assumptions of this question may be approximately valid only if the local shear layers associated with the walls and the fans occupy a very small fraction of the volume of the vessel.

(c) Starting from $dk/dt = -\varepsilon$, using $\varepsilon = k^{3/2}/L_0$, the initial condition $k = k_0$ at $t = 0$, and keeping L_0 constant, we get that $k^{-3/2}dk = -(1/L_0)dt$, whose solution is: $2L_0\left(k^{-1/2} - k_0^{-1/2}\right) = t$ (it can be left like this). However, if we further manipulate this expression we get: $k/k_0 = [1 + t/(2T_0)]^{-2}$, with $T_0 = L_0/\sqrt{k_0}$. So, at t equal to the initial eddy turnover time $T_0 = L_0/\sqrt{k_0}$, the k will have decayed to 0.44 of its initial value.

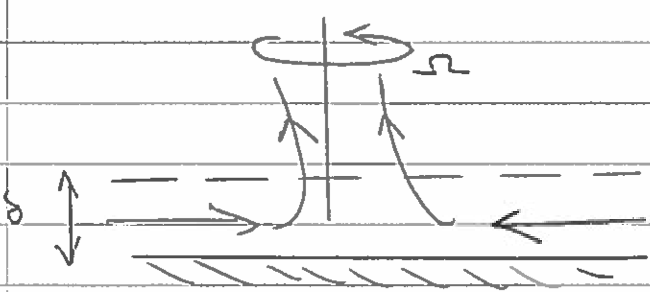
(d) From $k-\varepsilon$ model, assuming homogeneous flow (i.e. no gradients of any mean quantity in space), we get that $d\varepsilon/dt = -A\varepsilon^2/k$. If $k \sim t^{-m}$ and $\varepsilon \sim t^{-n}$, then $d\varepsilon/dt \sim t^{-n-1}$. For the lhs to be equal to rhs for all t, we must have: $t^{-n-1} \sim t^{2n}t^{-m}$, which means that $-n-1 = -2n+m$, hence $n = m+1$, which implies that $n = 5/2$.

3 (a) Karman layer



Fluid spins up near rotating surface. The zero radial pressure gradient in the still fluid outside the boundary layer is imposed on the layer, so there's nothing to match the centrifugal force in the rotating fluid. Fluid spins outward like in a centrifugal fan. The radial outflow must be matched by an axial inflow to satisfy continuity.

Bodewadt layer



Spinning fluid outside layer sets up radial pressure gradient

$$\frac{dp}{dr} = \rho\frac{(\Omega r)^2}{r} = \rho\frac{u_\theta^2}{r}$$

This radial pressure gradient is imposed on the boundary layer, where u_θ is lower. dp/dr is then greater than u_θ^2/r within the layer and drives an inflow towards the axis of rotation. By continuity, this must set up an axial flow away from the surface outside the layer.

(b) $\frac{u_\theta^2}{r}$ ~~balanced by radial shear stresses,~~

$$\Rightarrow \quad \frac{u_\theta^2}{r} \sim \nu\nabla^2 u_r \sim \nu\frac{\partial^2 u_r}{\partial z^2} \sim \nu\frac{u_r}{\delta^2}$$

$$\Rightarrow \quad \delta^2 \sim \nu\frac{u_r}{u_\theta^2/r} \sim \nu\frac{u_r}{u_\theta}\frac{1}{\Omega}$$

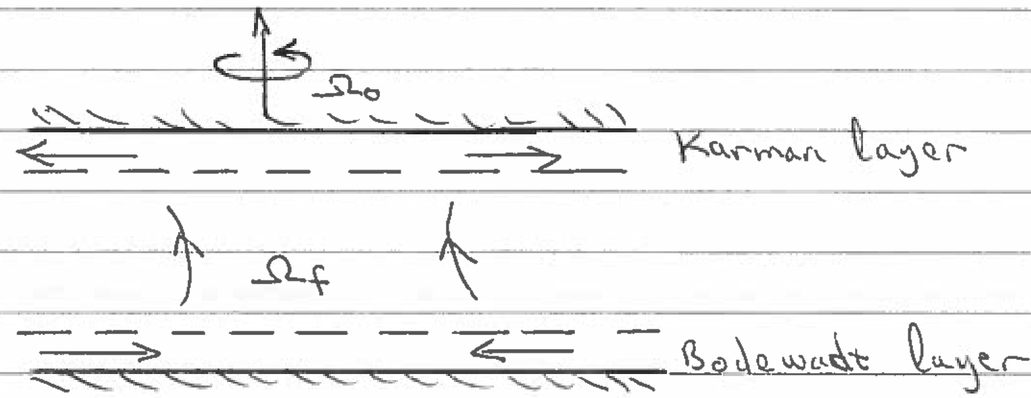
Observations show $u_r \sim u_\theta$, so $\delta \sim \sqrt{\nu/\Omega}$

Continuity: $(2\pi r)\delta u_r \sim \pi r^2 |u_z|$

$\Rightarrow \quad (2\pi r)\delta(\Omega r) \sim \pi r^2 |u_z|$

$\Rightarrow \quad |u_z| \sim \delta\Omega \sim \sqrt{\nu/\Omega}\,\Omega \sim \underline{\underline{\sqrt{\nu\Omega}}}$

(c)



Karman layer: $|u_z| \approx 0.9\sqrt{\nu(\Omega_0 - \Omega_f)}$

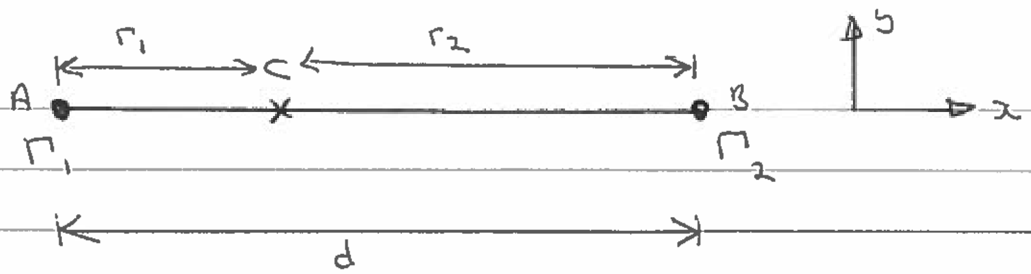
Bodewadt layer: $|u_z| \approx 1.4\sqrt{\nu\Omega_f}$

Continuity: $0.9\sqrt{\nu(\Omega_0-\Omega_f)} \approx 1.4\sqrt{\nu\Omega_f}$

$\Rightarrow \quad \Omega_0 - \Omega_f \approx (1.4/0.9)^2\,\Omega_f$

$\Rightarrow \quad \underline{\underline{\Omega_f \approx 0.29\,\Omega_0}}$

4 (a)



At A : $u_{yA} = -\dfrac{\Gamma_2}{2\pi d}$

At B : $u_{yB} = +\dfrac{\Gamma_1}{2\pi d}$

Consider : $r_1 = \dfrac{\Gamma_2 d}{\Gamma_1+\Gamma_2}$, $r_2 = \dfrac{\Gamma_1 d}{\Gamma_1+\Gamma_2}$

$(r_1 + r_2 = d)$

Relative to C rotation rate of A is (anticlockwise)

$$\Omega_A = \frac{u_y A}{r_1} = \frac{\Gamma_2}{2\pi d}\,\frac{\Gamma_1+\Gamma_2}{\Gamma_2 d} = \underline{\underline{\frac{\Gamma_1+\Gamma_2}{2\pi d^2}}}$$

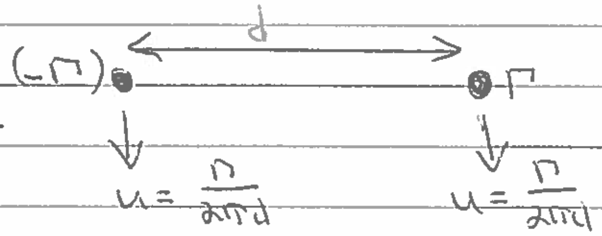
Relative to C rotation rate of B is (anticlockwise)

$$\Omega_B = \frac{u_y B}{r_2} = \frac{\Gamma_1}{2\pi d}\,\frac{\Gamma_1+\Gamma_2}{\Gamma_1 d} = \underline{\underline{\frac{\Gamma_1+\Gamma_2}{2\pi d^2}}} = \Omega_A$$

Thus A and B rotate about C at $\underline{\underline{(\Gamma_1+\Gamma_2)/2\pi d^2}}$

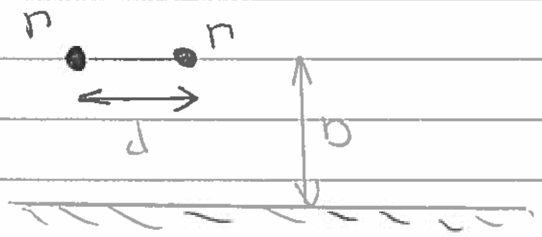
(b) If $\Gamma_1 = -\Gamma_2$ the $\Omega = 0$. Vortices propagate

as a pair at $\underline{\underline{u = \dfrac{\Gamma}{2\pi d}}}$

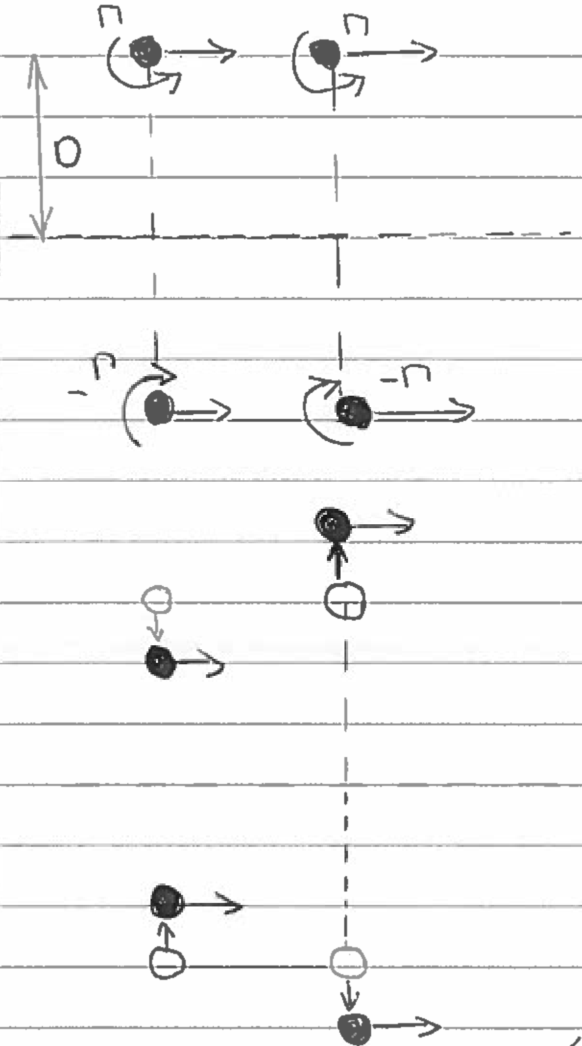


(Uniform translation)

(c)



To accomodate boundary, we method of images.



Each pair (+Γ, −Γ) wants to propagate to the right at speed

$$u = \frac{\Gamma}{4\pi D}$$

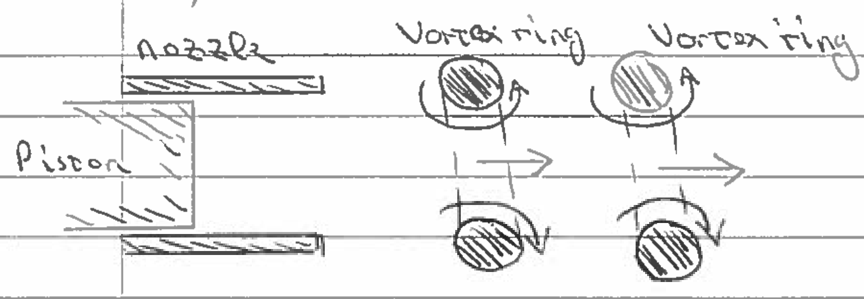
However, front pair tend to open up due to velocity from rear pair, while rear pair close up due to velocity of front pair.

Speed of front pair then falls while that of rear pair increases in accordance with

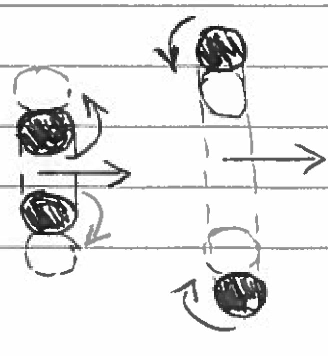
$$u = \frac{\Gamma}{4\pi \text{ separation}}$$

The rear pair then passes through the front pair, as in the "leap frog" of two vortex rings.

(d)



Create a pair of vortex rings (co-axial) using a piston and nozzle.



Rear vortex ring contracts and speeds up while front ring expands and slows down, as in part (c).

Rear vortex ring then passes through front ring.

7/7