4A12 2018 Crib

4A12 Turbulence 2018

1(a) V (volume) = 30x30x10=9000m^3. ρ (density) = 1.2 kg/m^3. Power per kg = turbulent kinetic energy dissipation per kg, $\Rightarrow$ Power needed = ρ*V*u^3/L$_t$ = 2.7kW.

(b) The typical diffusivity in this turbulent flow will be of order D=0.1*u*L, i.e. 0.025 m^2/s. At time t, a diffusion process advances over a distance $\delta \sim (D^*t)^{1/2}$, so $t \sim \delta^2/D$, hence if δ=10m, the time needed for the smoke to reach the ceiling will be of order 4000 s (66 min). This numerical estimate can be improved by starting from the long-time diffusion result of Taylor, which gives that $\delta^2=2u^2T_L{}^*t$, with T_L the Lagrangian integral timescale. We may assume that $T_L=L_t/u$. With this approach, the time needed becomes 200s, which is more realistic. (Examiners comment: both estimates are acceptable; but full marks only to those who used Taylor's result.) In real life, the smoke reaches the detector through buoyant convection as well as turbulent diffusion, which reduces the detection time significantly. Note also that the assumption of a long-time turbulent diffusivity is a good one, since we are dealing with lengthscales of the diffusion process >> L_t.

(c) In homogeneous isotropic decaying turbulence, dk/dt=$-\varepsilon$, with k=3/2u^2 and ε=u^3/L$_t$. If we put a=u/u$_0$ (u$_0$ being the initial turbulent intensity), $T_0=L_t/u_0$ (the initial integral timescale), and $\tau=t/T_0$, the governing equation becomes $da^2/d\tau=-2/3a^3$. The solution is $u/u_0=[1+t/(3T_0)]^{-1}$. So the turbulent intensity will fall to 10% of its initial value at 3T$_0$, i.e. at 3s.

2(a) Self-similar implies a state where the mean flow and the turbulence reach an equilibrium, where the turbulence adapts to the small changes of the mean flow and all radial profiles (e.g. of the mean velocities and the Reynolds stresses) assume the same shape when normalised by the centreline values. These centreline values change with downstream distance. The flow is characterised by a single lengthscale (e.g. of the order of the jet width) and a single velocity scale (the mean centreline velocity). The Reynolds stresses at the centreline are constant factors of the mean velocity (squared).

(b) At any distance x from the origin, the jet axial velocity profile is $U(r)=U_c(x)F(\eta)$, $\eta=r/\delta(x)$, $F(\eta)$ is the self-similar function describing the profile, $\delta(x)$ is the characteristic width, and $U_c(x)$ is the characteristic velocity at the centreline. The momentum flow rate M is conserved, hence

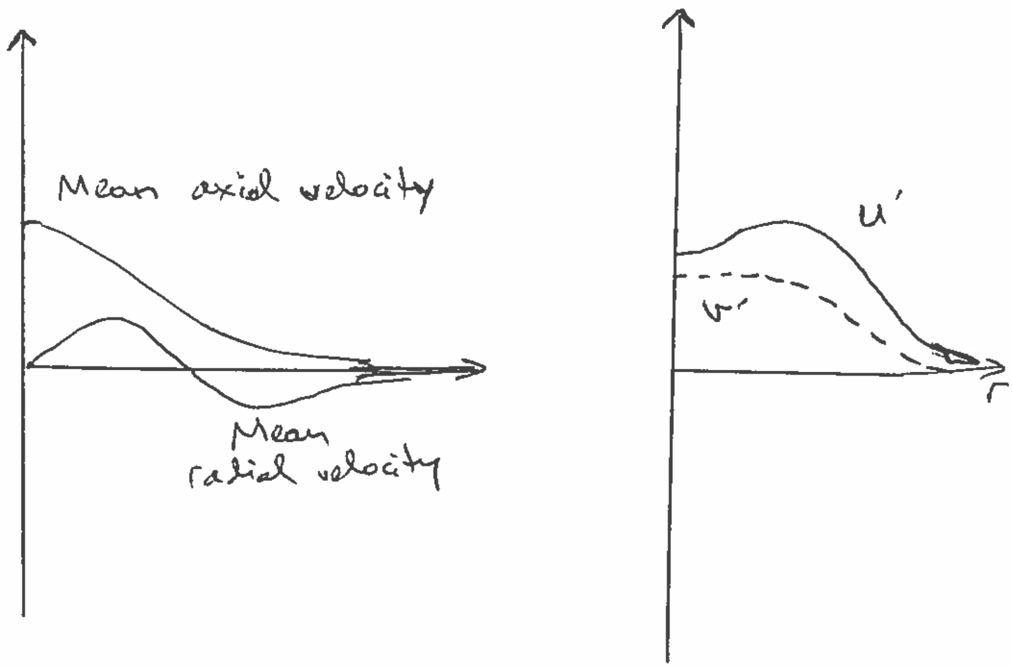
$$M=\int_0^\infty U(r)U(r)2\pi r dr = 2\pi U_c(x)U_c(x)\delta^2(x)\int_0^\infty F^2(\eta)d\eta$$

is constant for all x. The integral is not a function of x, so since in the axisymmetric jet $\delta(x)\sim x$, the characteristic velocity $U_c(x)\sim x^{-1}$.

(c) Similarly for the scalar, if it obeys a self-similar distribution, it must obey $c(r)=C(x)G(\eta)$. Since the scalar is conserved across the jet, its mass flow rate must stay constant. Hence, the integral below must be independent of x.

$$I=\int_0^\infty U(r)c(r)2\pi r dr = 2\pi U_c(x)C(x)\delta^2(x)\int_0^\infty F(\eta)G(\eta)d\eta$$

which implies that the characteristic scalar at the centreline $C(x) \sim x^{-1}$. The reduction in the value of the scalar at the centreline implies mixing with the ambient fluid. This comes about due to the entrainment (the pressure in the jet drops slightly, hence the ambient higher pressure "pushes" fluid radially into the jet). Alternatively, entrainment can be understood as the effect of the turbulence at the edges of the jet to "eat" into the irrotational ambient fluid. Once ambient fluid has been entrained, it mixes with the jet fluid through the action of the intense turbulence in the jet and the scalar concentration decreases. The radial profiles of the mean velocity components and the Reynolds stresses are shown below.



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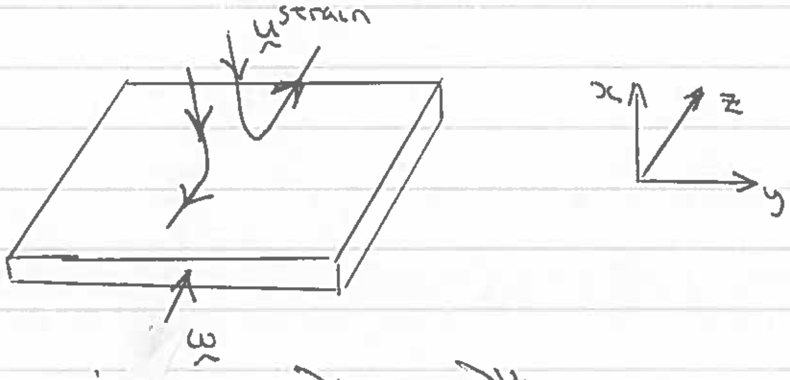
(3) (a)



$$\nabla \cdot \underset{\sim}{u}^{strain} = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = -\alpha + \alpha = 0$$

$$\nabla \times \underset{\sim}{u}^{strain} = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} = 0$$

(b) Φ (per unit length) $= \int_{-\infty}^{\infty} \omega_z \, dx = \int_{-\infty}^{\infty} \frac{\Phi}{\sqrt{\pi}\, \ell} \exp\left[-\left(\frac{x}{\ell}\right)^2\right] dx$

$$= \frac{\Phi}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \, dx$$

$$= \Phi \quad \checkmark\checkmark$$

Φ is conserved because neither advection nor diffusion can change the flux of vorticity and according to Helmholtz's laws stretching does not change the flux.

(c) Steady flow: $(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{\omega} = (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u} + \nu \nabla^2 \underset{\sim}{\omega}$

$$(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{\omega} = -\alpha x \frac{\partial \omega_z}{\partial x} \hat{\underset{\sim}{e}}_z = \frac{2\alpha x^2}{\ell^2} \underset{\sim}{\omega}$$

$$(\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u} = \omega \frac{\partial}{\partial z}(u_z \hat{\underset{\sim}{e}}_z) = \alpha \underset{\sim}{\omega}$$

$$\nu \nabla^2 \underset{\sim}{\omega} = \nu \frac{\partial^2 \omega}{\partial x^2} = \nu \frac{\partial}{\partial x}\left[-\frac{2x}{\ell^2}\underset{\sim}{\omega}\right] = \nu\left[\frac{4x^2}{\ell^4} - \frac{2}{\ell^2}\right]\underset{\sim}{\omega}$$

For steady flow

$$\frac{2\alpha x^2}{l^2} = \alpha + \frac{2\nu}{l^2}\left[\frac{2x^2}{l^2} - 1\right]$$

$$\Rightarrow \quad \alpha\left[\frac{2x^2}{l^2} - 1\right] = \frac{2\nu}{l^2}\left[\frac{2x^2}{l^2} - 1\right]$$

$$\Rightarrow \quad \underline{\underline{l = \sqrt{2\nu/\alpha} = l_0}}$$

(d) There is a balance between inward advection of $\underset{\sim}{\omega}$ by u_x, stretching of $\underset{\sim}{\omega}$ by u_z and outward diffusion of $\underset{\sim}{\omega}$.

(e) For $l > l_0$ diffusion is too weak to combat the inword advection of $\underset{\sim}{\omega}$. So the vortex sheet thins untill $l(t)$ reaches l_0.

For $l < l_0$ diffusion is too intense to be balanced by the inward advection of $\underset{\sim}{\omega}$. So the vortex sheet thickens until $l(t)$ reaches l_0.

(4) (a) δ can only depend on ν and Ω as there are no other variables.

$P = 3$, $D = 2$ (length, time) $\Rightarrow G = P - D = 1$

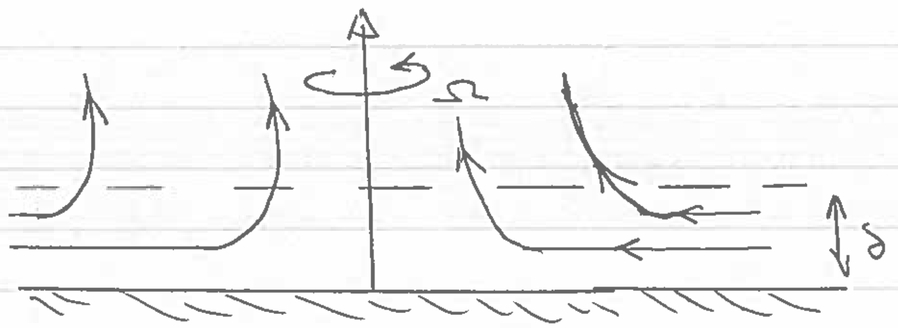
Can only make 1 dimensionless group from δ, ν and Ω.

δ: m, ν: m^2/s, Ω: s^{-1}

By inspection, dimensionless group is $\dfrac{\delta}{\sqrt{\nu/\Omega}}$

$$\Rightarrow \quad \underline{\underline{\delta \sim \sqrt{\nu/\Omega}}}$$

(b)



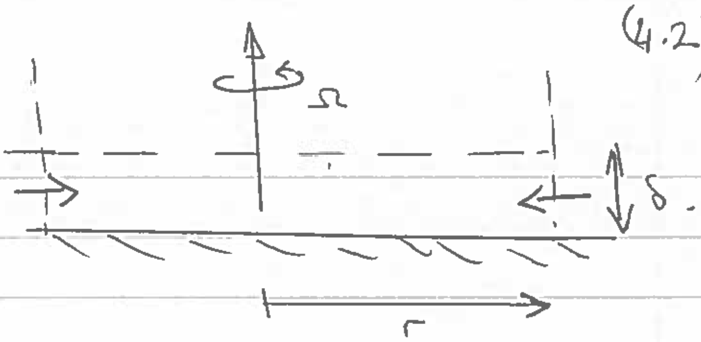
The radial inflow arises from the radial pressure gradient that is established by the balance <u>outside</u> the boundary layer

$$\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2}{r} = \rho \Omega^2 r \quad \text{[external force balance]}$$

Because the centrifugal force is weaker in the boundary layer (u_θ is smaller) the radial pressure gradient exceeds the centrifugal force and drives a radial inflow.

Continuity of mass then requires an axial flow away from the surface.

(c) Use continuity:



$$\text{Inflow} = 2\pi r\, u_r \delta$$

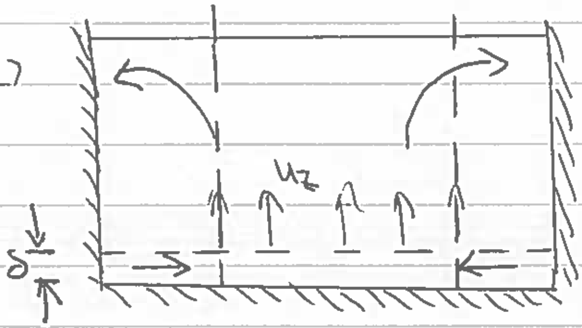
$$\sim 2\pi r\,(\Omega r)\sqrt{\nu/\Omega}$$

$$\sim \pi r^2 \sqrt{\nu\Omega} \quad = \pi r^2 u_z \quad \Rightarrow \underline{\underline{u_z \sim \sqrt{\nu\Omega}}}$$

Or, $\dfrac{\partial u_z}{\partial z} = -\dfrac{1}{r}\dfrac{\partial}{\partial r}(r u_r) \sim +\dfrac{1}{r}\dfrac{\partial}{\partial r}(\Omega r^2) \sim \Omega$

$\Rightarrow \underline{\underline{u_z \sim \Omega\delta \sim \sqrt{\nu\Omega}}}$ (independant of r)

(d) (i)



Use annular control volume, depth H

$u_r^{core} H \sim u_r\delta \sim (\Omega r)\delta$

(outflow) (inflow)

in boundary layer

(ii) $u_r^{core} \sim \dfrac{\Omega\delta r}{H} \sim \dfrac{\sqrt{\nu\Omega}}{H}\, r$

$$\frac{\partial u_z^{core}}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{core}\right) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sqrt{\nu\Omega}}{H} r^2\right)$$

$$\Rightarrow \frac{\partial u_z^{core}}{\partial z} \sim -\frac{\sqrt{\nu\Omega}}{H}$$

to satisfy $u_z = 0$ at $z = H$

$$\Rightarrow \underline{\underline{u_z^{core} \sim \sqrt{\nu\Omega}\left[1 - \frac{z}{H}\right]}}$$

(iii) Spin down time is time taken to flush the contents of tank through viscous Bödewadt layer, which is turn-over time of u^{core}

$$\text{Spin down time} \sim \frac{H}{u_z^{core}} \sim \underline{\underline{\frac{H}{\sqrt{\nu\Omega}}}}$$

Question 1: Turbulent Mixing
A popular, standard question, answered very well in the most part. There was one difficult part on the smoke dispersion that was answered well by only very few students; the fact that Taylor's dispersion was not asked regularly in the past may have contributed to this.

Question 2: Self-similarity applied to turbulent jet
A significant part of this question was straight out of the lecture notes, but it was still not done very well. The students intuition on where is the turbulence produced across the jet was broadly correct.

Question 3: Vortex Stretching
Nearly all students attempted this question and most answered it well. Some students had problems with the physical interpretation of part (e).

Question 4: Ekman Pumping
All students attempted this question and the overall performance was very good. Most mistakes occurred in parts d(ii) and d(iii), where the spin-down time was required.