4A15

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1. (a)

$$\Box^2 p' = \frac{\partial Q}{\partial t}$$

From the data card :

$$g(\underline{x}, t \mid \underline{y}, \tau) = \frac{\delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\}}{4\pi c_0 |\underline{x}-\underline{y}|}$$

$$\Rightarrow p' = g * \frac{\partial Q}{\partial t}$$

$$p' = \int_{-\infty}^{\infty}\int_{0}^{t} \frac{\delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\}}{4\pi c_0 |\underline{x}-\underline{y}|} \frac{\partial Q}{\partial \tau}\, d\underline{y}\, d\tau$$

The source (charge) is a point source. Let the source be at the origin.

$$\Rightarrow \frac{\partial Q}{\partial t} = \frac{\partial q}{\partial t}(t)\,\delta(\underline{x}) \quad . \qquad -(1)$$

$$\Rightarrow p' = \int_{-\infty}^{\infty}\int_{0}^{t} \frac{\delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\}}{4\pi c_0 |\underline{x}-\underline{y}|} \frac{\partial q}{\partial \tau}(\tau)\,\delta(\underline{y})\, d\tau\, d\underline{y}$$

$$= \int_{0}^{t} \frac{\delta\{x - c_0(t-\tau)\}}{4\pi c_0 x} \frac{\partial q}{\partial \tau}(\tau)\, d\tau$$

where $x = |\underline{x}|$.

$$\therefore \rho' = \frac{1}{4\pi c_0^2 x} \frac{\partial q}{\partial t}\left(t - \frac{x}{c_0}\right) , \quad t > \frac{x}{c_0} \qquad (2)$$

$$= 0 \qquad \text{otherwise.}$$

$$p' = c_0^2 \rho' \qquad (3)$$

$$\Rightarrow p' = \frac{1}{4\pi x} \frac{\partial q}{\partial t}\left(t - \frac{x}{c_0}\right) , \quad t > \frac{x}{c_0}$$

At $x = r_0$, we get

$$p'(r_0, t) = \frac{1}{4\pi r_0} \frac{\partial q}{\partial t}\left(t - \frac{r_0}{c_0}\right) , \quad t > \frac{r_0}{c_0}$$

From the given form of solution,

$$\frac{1}{4\pi r_0} \frac{\partial q}{\partial t}\left(t - \frac{r_0}{c_0}\right) = A e^{-\frac{1}{t_0}(t - r_0/c_0)} \frac{\left[t_0 - \left(t - \frac{r_0}{c_0}\right)\right]}{t_0^2}$$

$$\Rightarrow \frac{\partial q}{\partial t}(t) = 4\pi r_0 A e^{-t/t_0} \frac{[t_0 - t]}{t_0^2} \qquad (4)$$

From (1)

$$\dot{Q} = \dot{q}\,\delta(\underline{x})$$

$$\Rightarrow \boxed{\dot{Q} = 4\pi r_0 A\, e^{-t/t_0} \frac{[t_0 - t]}{t_0}\, \delta(\underline{x})}$$

(b) From (2), (3) & (4), the pressure field at any distance r is given by

$$p'(r,t) = \frac{1}{\cancel{4\pi} r} \cancel{4\pi} r_0 A\, e^{-\frac{1}{t_0}(t - r/c_0)} \left(\frac{t_0 - t - \frac{r}{c_0}}{t_0}\right) \qquad t > \frac{r}{c_0}$$

$$= 0 \qquad t < \frac{r}{c_0}$$

(c) For $r \gg 1$

$$u' = \frac{p'}{\rho_0 c_0}$$

$$I = p' u' = \frac{p'^2}{\rho_0 c_0}$$

$$\therefore I = \begin{cases} \left(\frac{r_0}{r} \frac{A}{t_0} \right)^2 \frac{1}{\rho_0 c_0} \cancel{(t_0 - \tau)^2 e} \left[t_0 - (t-\tau) \right]^2 e^{-\frac{2}{t_0}[(t-\tau)]} & \text{for } t > \tau \\ 0 & t < \tau \end{cases}$$

$$\left(\tau = \frac{r}{c_0} \right)$$

The instantaneous power is given by

$$I \cdot 4\pi r^2$$

The total work done by the source is given by

$$W = \int_0^\infty I \, 4\pi r^2 \, dt$$

$$= \left(\frac{r_0 A}{t_0} \right)^2 \frac{4\pi}{\rho_0 c_0} \int_\tau^\infty \left[t_0 - (t-\tau) \right]^2 e^{-\frac{2}{t_0}(t-\tau)} \, dt$$

Let $t - \tau = T$

then
$$\int_{\tau}^{\infty} \left[t_0 - (t-\tau)\right]^2 e^{-\frac{2}{t_0}(t-\tau)} \, dt$$

$$= \int_{0}^{\infty} \left[t_0 - T\right]^2 e^{-\frac{2}{t_0}T} \, dT$$

$$= \frac{t_0^3}{4}$$

$$\therefore \boxed{W = \frac{r_0^2 A^2 \cancel{4}\pi t_0}{\rho_0 c_0 \cdot \cancel{t_0} \cancel{4}}}$$

2 (a) From the datacard

$$\Box^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$ 3

Using the Green's f^n for the wave-equation from the data card, the sound field from a single eddy is given by

$$\rho' = \int\int \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \frac{\delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\}}{4\pi c_0 |\underline{x}-\underline{y}|} d\underline{y}\, d\tau$$

In the far field,

$$\frac{1}{|\underline{x}-\underline{y}|} \sim \frac{1}{|\underline{x}|} = \frac{1}{x}$$ 1

$$\therefore \rho' \sim \frac{1}{4\pi c_0 x} \iint \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\} d\underline{y}\, d\tau.$$

Using a property of convolution integrals,

$$\rho' = \frac{1}{4\pi c_0 x} \frac{\partial^2}{\partial x_i \partial x_j} \iint T_{ij}\, \delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\} d\underline{y}\, d\tau.$$ 1

Using the property of $\delta()$ function to integrate with respect to τ,

$$\rho' = \frac{1}{4\pi c_0^2 x} \frac{\partial^2}{\partial x_i \partial x_j} \int T_{ij}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_0}\right) d\underline{y}$$

2

For a compact eddy, we can neglect retarded time differences. Therefore,

$$\rho' = \frac{1}{4\pi c_0^2 x} \frac{\partial^2}{\partial x_i \partial x_j} \int T_{ij}\left(\underline{y}, t - \frac{x}{c_0}\right) d\underline{y}$$

2

Using the hint:

$$\rho' = \frac{1}{4\pi c_0^2 x} \frac{1}{(c_0)^2} \left(\frac{x_i x_j}{x^2}\right) \frac{\partial^2}{\partial t^2} \int T_{ij}\left(\underline{y}, t - \frac{x}{c_0}\right) d\underline{y}$$

1

with $T_{ij} = \rho_0 u_i u_j$

Let the length scale of the eddy be l, velocity scale u', then

$$\rho' \sim \frac{1}{c_0^4 x} \beta_i \beta_j \frac{u'^2}{l^2} \rho_0 u'^2 l^3$$

$$= \rho_0 \beta_i \beta_j \left(\frac{l}{x}\right) M^4$$

2

b) The power radiated from a single eddy is

$$P \sim p'u'x^2 = \frac{p'^2}{\rho_0 c_0} x^2 = \frac{c_0^4 \rho'^2 x^2}{\rho_0 c_0}$$

3

$$\therefore P \sim \frac{c_0^3}{\cancel{\rho_0}} \cancel{x^2} \cdot \rho_0^{\cancel{2}} (\beta_i \beta_j)^2 \frac{l^2}{\cancel{x^2}} M^8$$

3

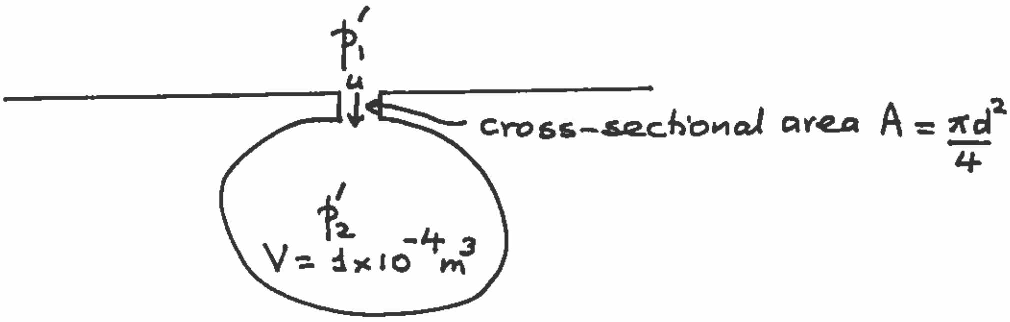
$$= \rho_0 (\beta_i \beta_j)^2 \frac{l^2}{c_0^5} u'^5$$

Assume that the eddies are randomly distributed, therefore, we loose directionality.
& u' scales as u_j and $l \sim d_j$

$$\therefore \boxed{P_{jet} = \rho_0 \frac{d_j^2}{c_0^5} u_j^8}$$

2

3



$$p_1' - p_2' = \rho_0 \ell \frac{du'}{dt} + \alpha u' \qquad (1)$$

The pressure rise in the cavity is related to the stored air.

$$p_2' = c_0^2 \rho_2' \quad \text{and} \quad A \rho_0 u' = V \frac{d\rho_2}{dt}$$

a) For sound of frequency ω

$$p_2' = \frac{c_0^2 A}{i\omega V} \rho_0 u'$$

Substituting in equation (1) gives $p_1' = u'\left(\rho_0 \ell i\omega + \frac{c_0^2 A \rho_0}{i\omega V} + \alpha\right)$ (2)

$\ell = 0.6\, d$ $\qquad c_0 = \sqrt{\gamma R \bar{T}} = 491.0\ ms^{-1}$

$A = \frac{\pi d^2}{4}$ $\qquad \rho_0 = \bar{p}/R\bar{T} = 0.5807\ kgm^{-3}$

Substituting the numbers into equation (2) gives

$$p_1' = u'\left(0.3484\, d i\omega + \frac{1.0995 \times 10^9 d^2}{i\omega} + k \times 285.1\right)$$

in mks units.

b) For resonance $\quad \rho_0 \ell i\omega = -\frac{c_0 A \rho_0}{i\omega V}$

$$0.6 d\, \omega = \frac{c_0^2 \pi d^2}{4 \omega V}$$

$$\cancel{\omega^2 = c_0}$$

$$d = \frac{4 \omega^2 V \times 0.6}{\pi c_0^2}$$

for $\omega/2\pi = 400\,Hz$, this gives $\underline{\underline{d = 2\ mm}}$

c i) The rate of sound absorption by a single hole

$$= \overline{p_1' u'} \, A$$

At 400 Hz $\quad u' = \dfrac{p_1'}{\alpha}$ from equation (2).

Hence rate of sound absorption by a single hole

$$= \frac{\pi d^2}{4} \frac{\overline{p_1'^2}}{\alpha} = \frac{\pi d^2}{4} \frac{1}{\rho_0 c_0 k} \overline{p_1'^2}$$

$$= \pi \times \frac{(2\times10^{-3})^2}{4} \frac{1}{0.5807 \times 491 \times 0.1} \overline{p_1'^2}$$

$$= 1.10 \times 10^{-7} \overline{p_1'^2}$$

(ii) The rate of sound absorption / unit area of liner

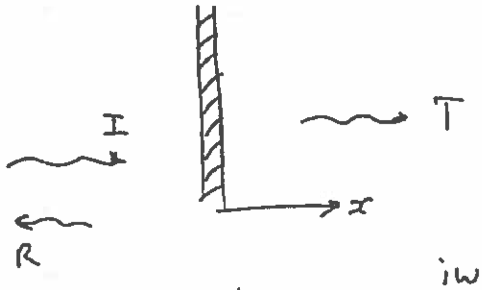
= rate of absorption per hole × number of holes/unit area

$$= 1.10\times10^{-7} \overline{p_1'^2} \times \frac{\text{open area ratio}}{\text{area of each hole}}$$

$$= 1.10 \times 10^{-7} \overline{p_1'^2} \times \frac{0.05}{\pi/4 \times (2^{-3})^2}$$

$$= 1.75 \times 10^{-3} \overline{p_1'^2}$$

4 (a)



Velocity potential $\phi = I e^{i\omega t - i\omega x/c_0} + R e^{i\omega t + i\omega x/c_0}$ $\quad x<0$

$\qquad\qquad\qquad\quad T e^{i\omega t - i\omega x/c_1}$ $\quad x>0$

Wall velocity $V e^{i\omega t}$

when $x = 0^-$ $\quad p = -i\omega \rho_0 (I+R)$

wall velocity $= \frac{i\omega}{c_0}(-I+R)$

when $x = 0^+$ $\quad p = -i\omega \rho_1 T$

wall velocity $= \frac{-i\omega}{c_1} T$

Continuity of normal velocity across $x=0$:

$\frac{i\omega}{c_0}(R-I) = -\frac{i\omega}{c_1} T = V \quad \Rightarrow \quad I - R = T\left(c_0/c_1\right)$ ①

Eq^n of motion of wall is $m\frac{dV}{dt} = p(x=0^-) - p(x=0^+)$

$\rightarrow mi\omega V = -i\omega\rho_0(I+R) + i\omega\rho_1 T \Rightarrow I+R = \left(\rho_1 + \frac{im\omega}{c_1}\right) T/\rho_0$ ②

① ÷ ② $\quad \frac{I+R}{I-R} = \frac{\rho_1 c_1 + im\omega}{\rho_0 c_0} = z_1 + i z_2 \qquad z_1 \equiv \frac{\rho_1 c_1}{\rho_0 c_0}$

$\qquad z_2 \equiv \frac{m\omega}{\rho_0 c_0}$

$\rightarrow \frac{R}{I} = \frac{z_1 + i z_2 - 1}{z_1 + i z_2 + 1}$

$$\left|R/I\right| = \left[\frac{(z_1-1)^2 + z_2^2}{(z_1+1)^2 + z_2^2}\right]^{1/2}.$$

But, require pressure reflection coefficient. However, this is the same since incident and reflected waves in same medium. ②

Hence, pressure reflection coefficient is given in ③

Depends on Z_1 = ratio of acoustic impedances in the 2 media and $Z_2 = \frac{m\omega}{\rho_0 c_0} \propto \frac{m}{\rho_0 \lambda}$, where λ is wavelength in $x<0$ $\therefore Z_2 \propto$ ratio of wall mass to fluid mass in one wavelength

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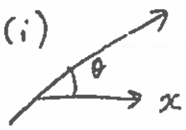
(b) (i) $\frac{\sin\theta}{c(x)} = \frac{\sin\theta_0}{c(0)}$ $\sin\theta = \frac{y'}{\sqrt{1+y'^2}}$



$\therefore \frac{y'}{(1+y'^2)^{1/2}} = \sin\theta_0 e^{x/l}$

$y'^2[1-\sin^2\theta_0 e^{2x/l}] = \sin^2\theta_0 e^{2x/l} \Rightarrow y' = \frac{\pm\sin\theta_0 e^{x/l}}{\sqrt{1-\sin^2\theta_0 e^{2x/l}}}$

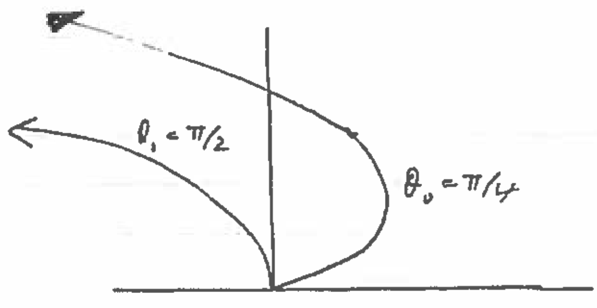
$\int dy = \int \frac{\sin\theta_0 e^{x/l}}{\sqrt{1-\sin^2\theta_0 e^{2x/l}}}$, $y = l\sin^{-1}[\sin\theta_0 e^{x/l}] + kl$

when $x=0$ $y=0$ $\Rightarrow k = -\sin^{-1}[\sin\theta_0]$

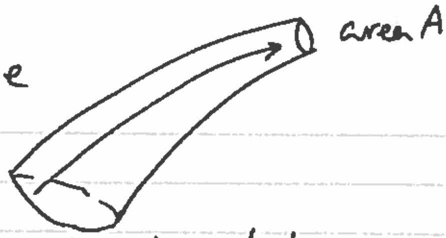
$\therefore y = l\sin^{-1}[\sin\theta_0 e^{x/l}] - l\sin^{-1}[\sin\theta_0]$

(ii) $\theta_0 = \pi/4$ y' is initially positive, but ray turns round once $\sin\theta_0 e^{x/l} = 1$

$\theta_0 = \pi/2$, y' becomes negative



(iii) Consider ray tube



energy is conserved

$\therefore$ $\overline{p'u'}$ × area is constant along tube

at high frequency assume plane-wave impedance

$$\therefore u' = \frac{p'}{\rho_0 c_0}$$

$$\therefore A \frac{\overline{p'^2}}{\rho_0 c_0} = \text{constant along tube}$$

$\therefore$ calculate A, know ρ_0 and c_0, hence find $\overline{p'^2}$.