4A15 crib 2019

1. (a)

The unsteady potential

$$\phi' = \frac{A}{r} e^{i\omega t - \frac{i\omega r}{c_0}}$$

$$\frac{\omega}{c_0} = k$$

$$\underline{v}' = \nabla\phi' = \frac{\partial \phi'}{\partial r}\underline{e_r}$$

$$= A\left(-\frac{ik}{r} - \frac{1}{r^2}\right) e^{i\omega t - ikr} \underline{e_r}$$

$$p' = -\rho_0 \frac{\partial \phi'}{\partial t} = = -i\omega \rho_0 \frac{A}{r} e^{i\omega t - ikr}$$

(b) In the farfield

$$\underline{v}' \sim -\frac{Aik}{r} e^{i\omega t - ikr}$$

$$\overline{I} = \frac{1}{2} \mathrm{Re}\{\hat{p}\hat{v}^*\}$$

$$= \frac{1}{2} \left\{ \frac{A^2}{r^2} \rho_0 \omega k \right\}$$

$$\text{Power} = \overline{I} \cdot 4\pi r^2$$

$$= 2\pi A^2 \rho_0 \omega k. \qquad \text{— (1)}$$

(c) $$|p'| = \frac{2A\rho_0\omega}{r}\left|\cos\left(ka\cos\theta + \frac{\psi}{2}\right)\right|$$

$$I = \frac{1}{2}\frac{|p'|^2}{\rho_0 c_0}$$

$$= \frac{1}{\cancel{2}}\,\cancel{4}^{2}\frac{A^2\rho_0^2\omega^2}{r^2}\cos^2\left(ka\cos\theta + \frac{\psi}{2}\right)$$

$$\text{Power} = \frac{2\pi r^2}{\rho_0 c_0}\int_0^{\pi}\frac{2A^2\rho_0^2\omega^2}{r^2}\cos^2\left(ka\cos\theta + \frac{\psi}{2}\right)\sin\theta\, d\theta$$

$$= \frac{4\pi A^2\rho_0\omega^2}{c_0}\int_0^{\pi}\cos^2\left(ka\cos\theta + \frac{\psi}{2}\right)\sin\theta\, d\theta$$

Let ~~$\sin\theta = u$~~ $\cos\theta = u$

$\Rightarrow$ ~~$\cos\theta\, d\theta = du$~~ $-\sin\theta\, d\theta = du$

$$\therefore \text{Power} = \frac{4\pi A^2\rho_0\omega^2}{c_0}\int_{-1}^{1}\cos^2\left(kau + \frac{\psi}{2}\right)du$$

Using the hint:

$$\text{Power} = \frac{4\pi A^2\rho_0\omega^2}{c_0}\left\{1 + \frac{1}{2ka}\cos\psi\,\sin(2ka)\right\}$$

— (2)

$\frac{(2)}{(1)}$ gives :

$$\frac{\cancel{4}^{2}\cancel{\pi}\cancel{A^2}\cancel{\rho_0}\omega^2}{c_0} \cdot \frac{1}{\cancel{2}\cancel{\pi}\cancel{A^2}\cancel{\rho_0}\omega k}\left\{1 + \frac{1}{2ka}\cos\psi\sin(2ka)\right\}$$

Using $\omega/k = c_0$,

$$= 2\left\{1 + \frac{1}{2ka}\cos\psi\sin(2ka)\right\}$$

(d) When $ka \gg 1$

The power from the two sorces is twice the power from a single source. There is no interference because, for $ka \gg 1$, the separation $(2a) \gg \lambda$, the the wavelength. The sources are independent and the power just adds up linearly.

When $ka \ll 1$

$$\frac{\sin(2ka)}{2ka} \sim 1$$

$$\therefore \quad \frac{\text{Power (2 sources)}}{\text{Power (1 source)}} = 2\left[1+\cos\psi\right]$$

When $\psi = 0$, the two sorces are in phase, the pressure field is doubled and the power becomes ~~four tim~~ four times the power from a ~~sigle~~ single source.

When $\psi = \pi$, the ~~t~~two sources are out of phase and cancel each other out.

2 (a) The conservation eq's can be written as

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \underline{u}' = 0 \qquad -(1)$$

$$\rho_0 \frac{\partial \underline{u}'}{\partial t} + \nabla p' = F \underline{e}_{x_1} \delta(x_1)\delta(x_2)\delta(x_3) \qquad -(2)$$

$\frac{\partial}{\partial t}(1) - \nabla.(2)$ gives:

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 \overbrace{p'}^{c_0^2 \rho'} = -\frac{\partial}{\partial x_1}\left\{F(t)\,\delta(x_1)\,\delta(x_2)\,\delta(x_3)\right\}$$

Using the Green's function:

$$\rho' = -\frac{\partial}{\partial x_1}\iiiint \frac{\delta(|\underline{x}-\underline{y}| - c_0(t-\tau))}{4\pi c_0\,(\underline{x}-\underline{y})} F(\tau)\,\delta(y_1)\,\delta(y_2)\,\delta(y_3)\, dy_1\, dy_2\, dy_3\, d\tau .$$

Carrying out the $\underline{y}$ integrals by using the property of the $\delta()$ function:

$$\int g(y)\,\delta(x-y)\,dy = g(x),$$

$$\rho'(\underline{x}, t) = -\frac{\partial}{\partial x_1}\int \frac{\delta(r - c_0(t-\tau))}{4\pi c_0 r} F(\tau)\, d\tau .$$

$$= -\frac{1}{4\pi c_0^2}\frac{\partial}{\partial x_1}\frac{F\left(t - \frac{r}{c_0}\right)}{r}$$

$$p' = c_0^2 \rho'$$

$$\therefore \; p'(\underline{x},t) = -\frac{1}{4\pi} \frac{\partial}{\partial x_1} \left(\frac{F(t - r/c_0)}{r} \right)$$

(b)

$$F(t) = \int_{-L}^{L} f(x_3, t)\, dx_3$$

$$= \int_{-L}^{L} \varepsilon\, e^{i\omega t} \cos\left(\frac{\pi x_3}{2L}\right) dx_3$$

$$= \varepsilon e^{i\omega t} \, \frac{2L}{\pi} \left[\sin\left(\frac{\pi x_3}{2L}\right) \right]_{-L}^{L}$$

$$\therefore \; F(t) = \frac{4\varepsilon L}{\pi} e^{i\omega t}$$

Using the result derived in part (a):

$$p'(\underline{x},t) = -\frac{1}{4\pi} \cdot \frac{4\varepsilon \cdot L}{\pi} \frac{\partial}{\partial x_1} \left\{ \frac{e^{i\omega(t - \frac{r}{c_0})}}{r} \right\}$$

$$= -\frac{\varepsilon}{\pi^2} \cdot L \left\{ \frac{1}{r} \cdot \left(-\frac{i\omega}{c_0}\right) \frac{\partial r}{\partial x_1} - \underbrace{\frac{1}{r^2} \frac{\partial r}{\partial x_1}}_{0} \right\} \times e^{i\omega(t - \frac{r}{c_0})}$$

(far field $\Rightarrow \frac{1}{r^2} \ll \frac{1}{r}$)

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$$\Rightarrow 2r \frac{\partial r}{\partial x_1} = 2x_1$$

$$\text{or} \quad \frac{\partial r}{\partial x_1} = \frac{x_1}{r}$$

$$\therefore \quad p'(\underline{x}, t) = + \frac{i\omega \varepsilon L x_1}{\pi^2 r^2 c_0} e^{i\omega(t - \frac{r}{c_0})}$$

(c) $$|p'| = \frac{\varepsilon}{\pi^2}\left(\frac{x_1}{r}\right) \cdot (kL)$$

where $k = \omega / c_0$.

$kL \ll 1$ (compactness ratio)

$\therefore |p'| \ll 1$

The oscillating string on its own does not radiate much sound. Its efficiency can be increased by:

1. Using a bridge to transfer the vibration energy of the string to the body of the guitar. A vibrating plate would be an effective source of sound.
2. By putting a resonant cavity close to the string, the strong near field of the dipole

would drive an unsteady mass flow out of the resonator, which would act as an efficient monopole source.

3

a) $\dot{p}_2 = -\frac{c_0^2}{V} m'(t)$ arises from the mass flow rate out of the bulb causing a rate of change of density in the bulb which for an isentropic fluctuation then gives this relationship for the change in pressure. For it to be true we required

1) the pressure in the bulb is uniform, i.e. p_2' is a function of t only. This requires that the linear size of the bulb is much smaller than c_0/ω and that the fluid in the bulb has negligible inertia which requires that the cross-sectional area of the bulb is much larger than that of the neck.

2) the perturbations need to satisfy $p_2' = c_0^2 \rho_2'$. This in turn requires than the perturbations are linear and isentropic, i.e. that there is negligible heat transfer and irreversible effects like friction.

3) $m'(t)$ needs to be the mass flow rate out of the bulb as well as the mass flow rate out of the neck. This requires negligible mass storage in the neck, i.e. the neck is short compared with c_0/ω.

$p_2' - p_1' = l \frac{\dot{m}}{A}$ arises from momentum balance across the neck.

It requires:

1) Linear perturbations

2) p_2' and p_1' uniform over the cross-section

3) No viscous forces

4) the fluid in the neck all moves with the same speed which requires negligible boundary layers and negligible compressible mass storage in the neck, i.e. the neck is short compared with c_0/ω (as in 3) above). [30%]

b) For frequency ω $\quad p_2' = -\frac{c_0^2}{V i\omega} m'$

$$p_2' - p_1' = \frac{l i\omega}{A} m'$$

Eliminating p_2' between the two equations gives

$$-\frac{c_0^2}{V i\omega} m' - p_1' = \frac{l i\omega}{A} m'$$

$$p_1' = -\left(\frac{c_0^2}{V i\omega} + \frac{l i\omega}{A}\right) m' = \frac{l}{A i\omega}(\omega^2 - \omega_0^2) m'$$

where $\omega_0^2 = \frac{c_0^2 A}{V l}$ [15%]

c) $c_0/\omega = 0.425$ which is larger than the physical dimensions of the bottle. So the bottle can be treated as a point monopole with mass flow rate $m'(t)$ and $m'(t)$ is related to $p_1'(t)$ as in part b.

The sound power radiated from a point monopole $= 4\pi r^2 \overline{p'u'}$

In the far-field $u' = p'/\rho_0 c$ and from the hint $p' = \frac{\dot{m}(t - r/c_0)}{4\pi r}$.

Hence radiated sound power $= \frac{4\pi r^2}{\rho_0 c_0} \overline{p'^2(r,t)}$

$$= \frac{4\pi r^2}{\rho_0 c_0} \frac{\overline{\dot{m}^2}}{(4\pi r)^2} = \frac{\omega^2 \overline{m'^2}}{\rho_0 c_0 4\pi}$$

Substituting for m' from part b) gives

$$\text{Radiated sound power} = \frac{\omega^2}{\rho_0 c_0 4\pi} \overline{p_1'^2} \left| \frac{A i \omega}{\ell(\omega^2 - \omega_0^2)} \right|^2$$

$$= \frac{1}{\rho_0 c_0 4\pi} \frac{\hat{p}_1^2}{2} \left(\frac{A\omega^2}{\ell(\omega^2 - \omega_0^2)} \right)^2$$

from mean-square

$$= \frac{1}{1.2 \times 340 \times 4\pi} \frac{4^2}{2} \left(\frac{3 \times 10^{-4} \; 800^2}{5 \times 10^{-2} (800^2 - \omega_0^2)} \right)^2$$

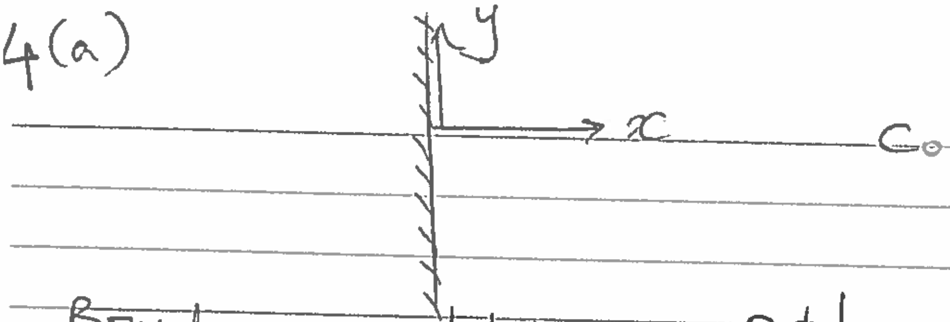
where $\omega_0 = 340 \sqrt{\dfrac{3 \times 10^{-4}}{1 \times 10^{-3} \times 5 \times 10^{-2}}}$

$= 832.83$ radians/sec

[65%]

$$= \underline{\underline{8 \times 10^{-6} \text{ Watts}}}$$

4(a)



Boundary condition $\left.\dfrac{\partial\phi}{\partial y}\right|_{x=0} = V_0\, e^{i\omega t - ik_2 y}$

Should be dphi/dx rather than dphi/dy —>

Seek solution $\phi = f(x)\, e^{i\omega t - ik_2 y}$

$$\nabla^2\phi = \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} \longrightarrow f'' - k_2^2 f = -\frac{\omega^2}{c_0^2} f$$

$$\rightarrow \quad f'' + \left(\frac{\omega^2}{c_0^2} - k_2^2\right) f = 0$$

$k_2 < \omega/c_0 \Rightarrow$ wave-like solution, out-going wave

$$\therefore \phi = A e^{i\omega t - ik_2 y - ik_1 x} \qquad k_1 = +\sqrt{\frac{\omega^2}{c_0^2} - k_2^2}$$

B.C. $\Rightarrow$ $-ik_2 A = V_0$, $A = \dfrac{iV_0}{k_2}$

$$\therefore \phi = \frac{iV_0}{k_2} e^{i\omega t - ik_1 x - ik_2 y}$$

$$p = -\rho_0 \frac{\partial\phi}{\partial t} = -i\rho_0\omega\phi = \underline{\underline{\frac{\rho_0 \omega V_0}{k_2} e^{i\omega t - ik_1 x - ik_2 y}}}$$

$k_2 > \omega/c_0 \Rightarrow$ evanescent wave

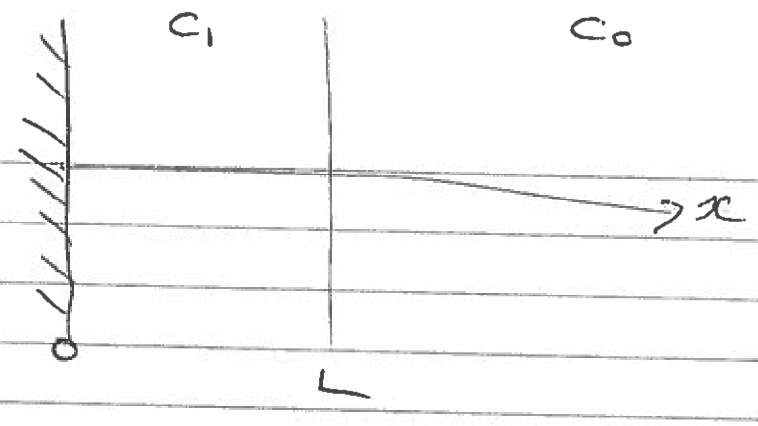
$$\phi = A e^{i\omega t - ik_2 y - k_1 x} \qquad k_1 = +\sqrt{k_2^2 - \omega^2/c_0^2}$$

now $\phi = \dfrac{iV_0}{k_2} e^{i\omega t - ik_2 y - k_1 x}$

$$p = \underline{\underline{\frac{\rho_0 \omega V_0}{k_2} e^{i\omega t - ik_2 y - k_1 x}}}$$

[8]

(b)



The phase speed of the wavy wall is $\left(\frac{k_2}{\omega}\right)^{-1}$

The condition $\frac{\omega}{k_2} > c_0$ means this wave is <u>supersonic</u> with respect to medium in $x > L$

The condition $\frac{\omega}{k_2} < c_1$ means this wave is <u>subsonic</u> with respect to medium in $0 < x < L$.

[4]

In $x > L$ have $\phi = A e^{i\omega t - ik_2 y - ik_1 x}$

where $k_1 = \sqrt{\frac{\omega^2}{c_0^2} - k_2^2}$

In $0 < x < L$ have

$$\phi = B e^{i\omega t - ik_2 y - \kappa x} + C e^{i\omega t - ik_2 y + \kappa x}$$

where $\kappa = \sqrt{k_2^2 - \frac{\omega^2}{c_1^2}}$

Wall b.c.

$$-ik_2 B - ik_2 C = V_0 , \quad B + C = \frac{iV_0}{k_2} \qquad (1)$$

Continuity of pressure at $x = L$:

$$-i\rho_1\omega\left(Be^{-\varkappa L} + Ce^{+\varkappa L}\right) = -i\rho_0\omega A e^{-ik_1L}$$

$$\rightarrow \quad Be^{-\varkappa L} + Ce^{+\varkappa L} = \frac{\rho_0}{\rho_1} e^{-ik_1L} A \qquad (2)$$

Continuity of normal velocity at $x = L$

$$-\varkappa Be^{-\varkappa L} + \varkappa Ce^{+\varkappa L} = -ik_1 A e^{-ik_1L}$$

$$\rightarrow \quad Be^{-\varkappa L} - Ce^{+\varkappa L} = \frac{ik_1}{\varkappa} e^{-ik_1L} A \qquad (3)$$

$$(2) + (3) \Rightarrow \quad Be^{-\varkappa L} = Ae^{-ik_1L}\left\{\frac{\rho_0}{\rho_1} + \frac{ik_1}{\varkappa}\right\}$$

$$(2) - (3) \Rightarrow \quad Ce^{+\varkappa L} = Ae^{-ik_1L}\left\{\frac{\rho_0}{\rho_1} - \frac{ik_1}{\varkappa}\right\}$$

into (1) $\quad Ae^{-ik_1L}\left\{\left(\frac{\rho_0}{\rho_1} + \frac{ik_1}{\varkappa}\right)e^{\varkappa L} + \left(\frac{\rho_0}{\rho_1} - \frac{ik_1}{\varkappa}\right)e^{-\varkappa L}\right\} = \frac{iV_0}{k_2}$

$$A = \frac{iV_0 e^{ik_1L}}{k_2 \Delta} \qquad \Delta \equiv \left(\frac{\rho_0}{\rho_1} + \frac{ik_1}{\varkappa}\right)e^{\varkappa L} + \left(\frac{\rho_0}{\rho_1} - \frac{ik_1}{\varkappa}\right)e^{-\varkappa L}$$

$$B = e^{\kappa L}\left(\frac{\rho_0}{\rho_1} + \frac{i k_1}{\kappa}\right)\frac{i V_0}{k_2 \Delta},$$

$$C = e^{-\kappa L}\left(\frac{\rho_0}{\rho_1} - \frac{i k_1}{\kappa}\right)\frac{i V_0}{k_2 \Delta}.$$

[8]

Q1 Sound power and interference

A popular question. The first three parts were answered well by most candidates. The last part turned out to be the most difficult. Candidates were able to the paths but very few could explain the physical consequence of the result. Most did not explain the effect of compactness of a source on sound power radiation.

Q2 Sound radiation from a vibrating string

A popular question. The first part was answered well but most candidates but some got the answer correctly by making an approximation that was not necessary – it is not necessary to approximate the retarded time term to zero because of compactness – it is exactly zero because of Delta functions present in the source (see crib). Part (b) was answered very well by most candidates. For part (c), most candidates gave one reason (correctly), but missed another reason – the use of a bridge to vibrate the body of the guitar.

Q3 Helmholtz resonator

This was a popular question and most undergraduates showed a good understanding of Helmholtz resonators and the sound they generate.

Q4 The wavy wall problem

The number and quality of attempts at this question were somewhat disappointing. Of the candidates that did attempt the question, most were able to distinguish between subsonic and supersonic motion and identify the correct behaviour of the unsteady field, although there were mistakes in applying the wall boundary condition to determine the unknown coefficient. In the final part of the question, few candidates were able to correctly match between the two fluid layers and obtain the radiating field at infinity.