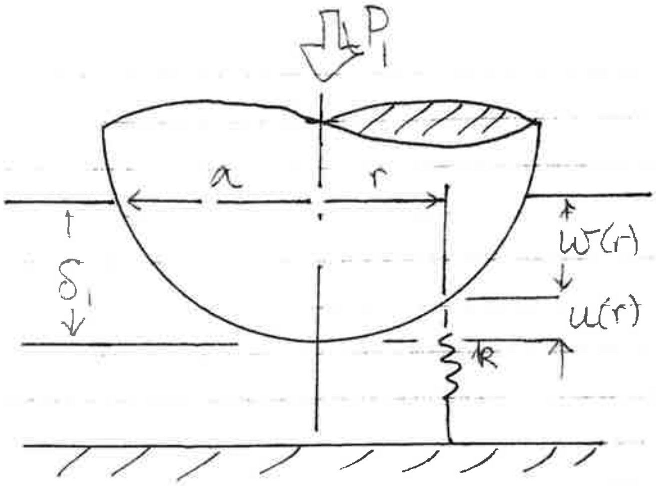
Q1



(a) Molecules at radius r are compressed by $w(r)$

where $w(r) = \delta_1 - u(r)$ and $u(r) = \dfrac{r^2}{2R}$

But $\delta_1 = \dfrac{a^2}{2R}$ $\therefore$ $w(r) = \dfrac{a^2 - r^2}{2R}$

If axial stiffness of each molecule is k, axial force $= k\,w(r)$

Since there are n molecules per unit area the effective pressure $p(r)$ at radius r $= k\,w(r).n$

i.e. $p(r) = \dfrac{nk}{2R}(a^2 - r^2)$

(b) Thus for equilibrium $P_1 = 2\pi \int_0^a p(r).r\,dr$

$$= 2\pi.\frac{nk}{2R}\int_0^a (a^2 r - r^3)\,dr$$

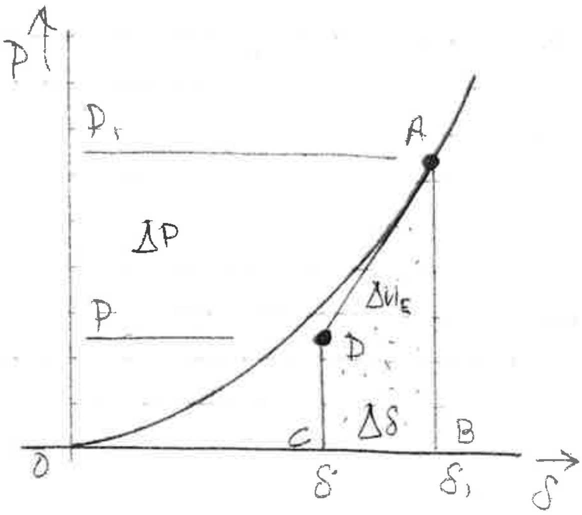
$$= \frac{\pi n k}{R} a^4 \left(\frac{1}{2} - \frac{1}{4}\right)$$

i.e. $P_1 = \dfrac{\pi n k a^4}{4R}$

But since $\delta_1 = a^2/2R$ $\quad P_1 = \pi n k R \delta_1^2$

(2)

(c) Thus plot of P vs. δ is as illustrated. Since all elastic, WD = energy stored = area under curve



i.e. $$U_{E_1} = \int_0^{\delta_1} P\,d\delta$$

$$= \pi n k R \int_0^{\delta_1} \delta^2\,d\delta$$

$$U_{E_1} = \frac{\pi n k R \delta_1^3}{3} \text{ or } \frac{\pi n k R}{3}\left(\frac{a^2}{2R}\right)^3$$

$$= \frac{\pi n k a^6}{24R^2}$$

(d) If SAM attached to indenter, when this withdraws by $\Delta\delta$, SAM acts as $(\pi a^2 n)$ springs acting in parallel. $\therefore$ $\Delta P = \pi a^2 n k \cdot \Delta\delta$

Thus system moves down A to D

(e) Then $U_E = U_{E_1} - \Delta U_E$ where ΔU_E = Area ABCD

$$U_E = \frac{\pi n k a^6}{24R^2} - \frac{1}{2}(P_1 + P)(\delta_1 - \delta)$$

$$\text{i.e. } \frac{\pi n k a^6}{24R^2} - \frac{1}{2}(P_1 + P) \cdot \frac{1}{\pi a^2 n k}(P_1 - P)$$

$$\Rightarrow \frac{\pi n k a^6}{24R^2} - \frac{1}{2\pi a^2 n k}(P_1^2 - P^2)$$

$$= \frac{\pi n k a^6}{24R^2} - \frac{1}{2\pi a^2 n k}\frac{\pi^2 n^2 k^2 a^8}{16R^2} - \frac{P^2}{2\pi a^2 n k}$$

$$\text{i.e. } U_E = \frac{\pi n k a^6}{96R^2} - \frac{P^2}{2\pi a^2 n k}$$

(f) If we write an expression for the total energy of the system U_{TOT} it must also account for the loss of p.e. of the machine U_M, and the change in surface energy arising from the adhesion, U_S. Both these terms will be −ve.

Hence $\underline{U_{TOT} = U_E(>0) + U_M(<0) + U_S(<0)}$

If each of these is written in terms of contact size a then $\left.\frac{\partial U_{TOT}}{\partial a}\right)_P = 0$ identifies a condition of instability. Now $U_M = -P\delta$ and $U_S = -\pi a^2 G$

where $G = \gamma_1 + \gamma_2 - \gamma_{12}$.

Hence $U_{TOT} = \frac{\pi}{96}\frac{nka^6}{R^2} - \frac{P^2}{2\pi nka^2} - \frac{Pa^2}{4R} - \pi a^2 G$

from which $P = \frac{\pi nka^4}{4R} - \pi(24nk)^{1/2}a^2$

"Pull-off" will occur when $\frac{dP}{da} \Rightarrow 0$

i.e. $a^4 = \frac{8R^2G}{nk}$

Substitution gives $\underline{P^o_{pulloff} = -2\pi RG}$ (tension)

This compares with the standard JKR result

of $\underline{P_{pulloff} = -\frac{3\pi}{2}RG}$

(a) For an electrode segment of length dx located at $x = x$, a parallel-plate approximation can be considered to express the electrostatic force (dF)

$$dF = \frac{\varepsilon_0 b V^2 dx}{2(d - \alpha x)^2}$$

The electrostatic torque can be calculated as:

$$M_e = \int_{a_1}^{a_2} x \, dF$$

$$= \int_{a_1}^{a_2} \frac{\epsilon_0 b V^2 x \, dx}{2(d - \alpha x)^2}$$

$$= \frac{\epsilon_0 b V^2}{2\alpha} \int_{a_1}^{a_2} \frac{(-d + \alpha x + d)\, dx}{(d - \alpha x)^2}$$

$$= \frac{\epsilon_0 b V^2}{2\alpha} \left\{ - \int_{a_1}^{a_2} \frac{dx}{(d - \alpha x)} + \int_{a_1}^{a_2} \frac{d \, dx}{(d - \alpha x)^2} \right\}$$

$$= \frac{\epsilon_0 b V^2}{2\alpha} \left\{ \left[\frac{\ln(d - \alpha x)}{\alpha} \right]_{a_1}^{a_2} + \frac{d}{\alpha} \left[\frac{1}{d - \alpha x} \right]_{a_1}^{a_2} \right\}$$

$$\therefore M_e = \frac{\epsilon_0 b V^2}{2\alpha^2} \left[\frac{d}{d - \alpha a_2} - \frac{d}{d - \alpha a_1} + \ln\left(\frac{d - \alpha a_2}{d - \alpha a_1} \right) \right]$$

(b) If $a_1 = 0$, $a_2 = l$ then:

$$M_e = \frac{\epsilon b V^2}{2\alpha^2}\left[\frac{d}{d-l\alpha} - 1 + \ln\left(\frac{d-l\alpha}{d}\right)\right]$$

$$M_e = \frac{\epsilon b V^2}{2\alpha^2}\left[\frac{l\alpha/d}{1-l\alpha/d} + \ln\left(1-\frac{l\alpha}{d}\right)\right] \quad \text{— (1)}$$

If $\theta = \alpha l/d << 1$ (small tilt angle)

$$M_e = \frac{\epsilon b V^2}{2\alpha^2}\left[\frac{\theta}{1-\theta} + \ln(1-\theta)\right]$$

$$= \frac{\epsilon b V^2}{2\alpha^2}\left[\theta(1+\theta+\theta^2+\dots) + (-\theta) - \frac{\theta^2}{2} - \frac{\theta^3}{3}\right]$$

$$= \frac{\epsilon b V^2 l^2}{2 d^2 \theta^2}\left[\frac{\theta^2}{2} + \frac{2\theta^3}{3} + \dots \text{H.O.T.}\right]$$

$$k_\theta \cdot \theta_{eq} \cdot d/l \simeq \frac{\epsilon b V^2 l^2}{2 d^2}\left[\frac{1}{2} + \frac{2}{3}\theta_{eq}\right]$$

$$\therefore \theta_{eq} \approx \frac{\dfrac{\epsilon b V^2 l^2}{4 d^2}}{\left[k_\theta d/l - \dfrac{\epsilon b V^2 l^2}{3 d^2}\right]}$$

(c) $$k_\theta \cdot \alpha_{PI} = \frac{\epsilon b V^2}{2\alpha_{PI}^2}\left[\frac{l\alpha_{PI}/d}{1-l\alpha_{PI}/d} + \ln\left(1-\frac{l\alpha_{PI}}{d}\right)\right]$$

and $$k_\theta = \left.\frac{\partial M_e}{\partial \alpha}\right|_{\alpha=\alpha_{PI}} = \frac{\epsilon b V^2}{2\alpha_{PI}^3}\left[\frac{-2l\alpha_{PI}/d}{1-l\alpha_{PI}/d} - \ln\left(1-\frac{l\alpha_{PI}}{d}\right) + \frac{(\alpha_{PI} l/d)^2}{(1-\alpha_{PI} l/d)^2}\right]$$

(d) when $\alpha_{PI} = 0.44\, d/l$ then from (1)

$$k_\theta \cdot 0.44\, d/l = \frac{\epsilon b V_{PI}^2}{2(0.44\, d/l)^2}\left[\frac{0.44}{1-0.44} + \ln(1-0.44)\right]$$

$$\therefore \quad V_{PF} = 0.91 \sqrt{\frac{k_\theta d^3}{\epsilon b l^3}}$$

Q3 (a)

$$x \approx \frac{F_e\, Q}{k_x}$$

$$F_e = \frac{N \epsilon_0 t}{g}\, v_{ac} \cdot V_{DC}$$

$$x \approx \frac{1000 \times 8.85\times10^{-12} \times 1 \times 10 \times 10 \times 10}{1 \times 10}$$

$$x \approx 0.885\ \mu m$$

(b) using CUED Mechanics databook,

$$\left|\frac{Y}{X}\right| = \frac{1}{\left\{\left[\left(1-\frac{\omega_x}{\omega_y}\right)^2\right]^2 + \left(\frac{2\zeta_y \omega_x}{\omega_y}\right)^2\right\}^{1/2}} \cdot \frac{2\Omega\,\omega_x}{\omega_y^2}$$

where $\omega_x = 10^5$ rad/s and $\omega_y = 1.1\times10^5$ rad/s.

$$\left|\frac{Y}{X}\right| = \frac{2\times 10 \times \frac{1}{1.1} \times \frac{1}{1.1\times10^5}}{16.53}$$ plugging in values

or $|Y| = 0.38$ nm

$\frac{\Delta C}{C}$ = fractional change in sense capacitance

$$= \frac{2|Y|}{g} = 0.76\times10^{-3}$$

(c) Thermomechanical noise $\overline{F_n} = \sqrt{4 k_B T\, b_y\, BW}$.

Equating to magnitude of Coriolis force in order to estimate the noise-equivalent rotation rate:

$$4 k_B T\, b_y\, BW = (2 m\, \omega_x\, x)^2\, \overline{\Omega_{zn}^2}$$

$$\therefore \overline{\Omega_{zn}^2} = \frac{k_B T\, b_y\, BW}{m^2\, \omega_x^2\, x^2}$$

Plugging in values :-

$$\bar{\Omega}_{zn} = 2.4 \times 10^{-3} \ \text{rad}/s/\sqrt{Hz}.$$

(d) For a differential parallel-plate scheme:

$$F_e \simeq \frac{1}{2} \epsilon_0 A V^2 \left[\frac{1}{(y_0 - y)^2} - \frac{1}{(y_0 + y)^2} \right]$$, y is the sense displacement

for one set of electrodes:

$$k_e = \frac{\partial F_e}{\partial y} = \epsilon_0 A V^2 \left[\frac{1}{(y_0 - y)^3} + \frac{1}{(y_0 + y)^3} \right].$$

If $y \ll y_0$ as seen from (b) then:

$$k_e \approx \frac{2 N \epsilon_0 A V^2}{y_0^3}$$

$\therefore$ $k_y - k_e = m \omega_x^2$ for mode matching

$\therefore$ $k_e = 2.1 \ N/m$

and $V = 1.54 \ V$ is the required tuning voltage.

Q4 (a)

$$\Delta P = kL$$

$$\frac{\Delta P}{Q} = \frac{12 \eta L}{h w^3} = R_F \quad (\text{flow resistance}).$$

Total flow resistance $= \dfrac{12\eta}{h}\left[\dfrac{L_1}{w_1^3} + \dfrac{L_2}{w_2^3} + \dfrac{L_3}{w_3^3}\right]$

where $L_1 = L_2 = 2\,mm$, $w_1 = w_2 = 100\,\mu m$

$L_3 = 1\,mm$, $w_3 = 10\,\mu m$.

$$\therefore R_F = \frac{12 \times 10^{-3}}{2 \times 10^{-4}}\left[\frac{20 \times 2}{(10^{-4})^2} + \frac{100}{(10^{-5})^2}\right]$$

$$\therefore Q = \frac{\Delta P}{R_F} = \frac{10^4}{60 \times 10^{12}}$$

$\therefore$ time taken to pump $1\,\mu l$ of solution

$$= \frac{10^{-9} \times 60 \times 10^{12}}{10^4} = 6 \text{ secs}.$$

(b) The channel resistance can be approximated by $R = \rho l / A$.

(i) $\therefore$ voltage drops in a 1:5:1 ratio down the length of the channel with a 10 V drop in the region prior to the channel constriction. $\therefore$ For this section:

$$|U_{01}| = \frac{80 \times 8.85 \times 10^{-12} \times 100 \times 10^{-3}}{10^{-3}} \times \frac{10}{2 \times 10^{-3}}$$

$$= 3.54 \times 10^{-4}\ m/s$$

$\therefore$ time taken $= \dfrac{2 \times 10^{-3}}{3.54 \times 10^{-4}} = 5.65$ secs.

(ii) Drift velocities given by $\mu_E E_x$

For leading species in initial part of the channel: $V_{tot} = U_{01} + \mu_{E1} E_x = 5.04 \times 10^{-4}$ m/s.

$\therefore$ time taken to enter constriction $= 3.97$ s.

and relative separation $= \Delta V \cdot 3.97 = 198.5\ \mu m$

In the constriction $|U_{03}| = 10 \times |U_{01}|$

$\therefore$ time taken for trailing species to exit constriction $= 4.41\text{ s} + 0.22\text{ s} = 4.63$ s.

In that time the leading species has travelled $= 0.463 \times 5.04 \times 10^{-4} = 233\ \mu m$ further down the channel and this is now the relative separation distance.

(iii) The species diffuse in solution such that the band size $= \sqrt{Dt}$, $t \propto L/u_0$

$\therefore$ use short columns, large electric fields and low ionic strength buffers to achieve large L_p and good separation while ensuring plug flow behaviour for transporting bulk electrolyte solution.

ENGINEERING TRIPOS PART IIB 2015
MODULE 4C15: MEMS DESIGN
ASSESSOR’S COMMENTS

Q1 AFM

Part (a) was generally well done but some students had difficulties with the calculations in parts (b) and (c). Poor sketches were produced for the graphs of load vs displacement in part (d). The qualitative argument to establish the pull-off criterion was generally well understood though not many candidates took this forward in terms of providing an estimate for the “pull-off” force in (e).

Q2 Torsional actuator

Many students struggled with the derivation of the electrostatic torque in part (a) even though a very similar example was covered in the lectures. Those who succeeded in establishing in a formulation for the electrostatic torque generally were able to do the rest of the question reasonably well with solutions indicating that most undergraduates had a good understanding of the electrostatic “pull-in” criterion.

Q3 Gyroscope

This question was generally popular and most students were able to establish the expression for the electrostatic force generated by a comb drive in part (a). However students had difficulty in part (b) in estimating the sense mode displacement though the necessary expression could be readily derived from the Mechanics databook. The formulation for thermomechanical noise in (c) was generally well done though students had some difficulty in estimating the electrostatic stiffness arising as a result of the parallel-plate sensing configuration in (d).

Q4 Microfluidics

This question was generally poorly done and less popular. There was one correct attempt for part (a) and none for part (b). The students did not appear to appreciate how the non-uniformity of the channel would impact the pressure distribution in (a) or the electrical field distribution in (b).