Engineering Tripos Part IIB: Module 4C2 Designing with Composites

Solutions - 2014/15 (M Sutcliffe)

1 **(a)** A laminate is made up of a stacked assembly of unidirectional plies, each having its fibre axis lying at a specified angle to a reference direction. They are used in preference to unidirectional plies because they are more isotropic, which can be desirable in some circumstances where loading is in more than one direction, or to avoid splitting failure.

A symmetric laminate is one possessing a mirror plane lying in the plane of the laminate i.e. the stacking sequence in the top half reflects that in the bottom half. A symmetric laminate does not exhibit bending-stretching coupling (the coupling stiffness [B]=0), i.e. in-plane loading will not generate any out-of-plane distortion and vice versa.

(b) (i)

$$\bar{S}_{16} = \left(2\,S_{11} - 2S_{12} - S_{66}\right)c^3 s - \left(2\,S_{22} - 2\,S_{12} - S_{66}\right)c\,s^3$$

where $c = \cos\theta,\ s = \sin\theta$

For $\overline{S_{16}}$ to be zero (no tensile-shear interactions):

$$\left(2\,S_{11} - 2S_{12} - S_{66}\right)c^3 s = \left(2\,S_{22} - 2\,S_{12} - S_{66}\right)c\,s^3$$

Two solutions are $\theta = 0^\circ$ ($s = 0$) and $\theta = 90^\circ$ ($c = 0$). [These directions commonly missed]. Third solution is:

$$\frac{s^2}{c^2} = \frac{2S_{11} - 2S_{12} - S_{66}}{2S_{22} - 2S_{12} - S_{66}} \Rightarrow \theta = \tan^{-1}\left(\left(\frac{2S_{11} - 2S_{12} - S_{66}}{2S_{22} - 2S_{12} - S_{66}}\right)^{1/2}\right)$$

$$S_{11} = \frac{1}{E_1} = \frac{1}{76}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.34}{76}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{5.5}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{2.3}$$

$$\theta = \tan^{-1}\left(\left(\frac{2S_{11} - 2S_{12} - S_{66}}{2S_{22} - 2S_{12} - S_{66}}\right)^{1/2}\right) = \tan^{-1}(2.53) = 68^{\circ}$$

$\overline{S_{16}}$ (& $\overline{S_{26}}$) are the “tensile-shear interaction” terms. An important feature of the off-axis loading of unidirectional laminae is the tensile-shear interactions which lead to distortions. Normal stresses induce shear strains and vice-versa. These can be minimised if the stacking sequence is chosen carefully, since the distortions from individual laminae may cancel each other out.

(ii) The lamina stress-strain of the epoxy-glass fibre composite lamina can be written

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{pmatrix} 0.2 \\ 0 \\ 0 \end{pmatrix}$$

Using the supplied values of the elastic constants of the composite, for $\theta = 60^\circ$

$$\bar{S}_{16} = \left(\frac{2}{76} + \frac{2\cdot 0.34}{76} - \frac{1}{2.3}\right)\cos^3 60 \sin 60 - \left(\frac{2}{5.5} + \frac{2\cdot 0.34}{76} - \frac{1}{2.3}\right)\cos 60 \sin^3 60$$

$= -0.023$ GPa^{-1}

Note: $\bar{S}_{16}$ can be of either sign (ie it can shear in either direction)

The shear strain is given by

$\gamma_{xy} = \bar{S}_{16}\sigma_x = 0.023\cdot 0.2 = 4.6\times 10^{-3}$ [or γ_{12}= 0.0377]

(c)

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \Rightarrow \nu_{21} = 0.03$$

Calculate [Q] in principal material axes (1, 2).

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = \frac{E_1}{1-0.3\times 0.03} = 1.009E_1$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = \frac{0.1E_1}{1-0.3\times 0.03} = 0.1009\,E_1$$

$$Q_{12} = \frac{\nu_{12}\,E_2}{1-\nu_{12}\nu_{21}} = \frac{0.3\times 0.1E_1}{1-0.3\times 0.03} = 0.03027E_1$$

$$Q_{66} = G_{12} = 0.05\,E_1 \qquad Q_{16} = Q_{26} = 0$$

$$\left[Q\right] = E_1 \begin{bmatrix} 1.009 & 0.03027 & 0 \\ 0.03027 & 0.1009 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}$$

Calculate the transformed stiffness matrix $[\bar{Q}]$ in the global x-y axes. The transformed stiffness matrix for the -45° ply (1st ply) is given by

$$\left(\overline{Q}_{11}\right)_{-45^\circ} = E_1\left[1.009\ c^4 + 0.1009\ s^4 + 2\left(0.03027 + 2\cdot 0.05\right)s^2c^2\right] = 0.34E_1$$
$$\left(\overline{Q}_{12}\right)_{-45^\circ} = E_1\left[\left(1.009 + 0.1009 - 4\cdot 0.05\right)s^2c^2 + 0.03027\left(c^4 + s^4\right)\right] = 0.24E_1$$
$$\left(\overline{Q}_{22}\right)_{-45^\circ} = E_1\left[1.009\ s^4 + 0.1009\ c^4 + 2\left(0.03027 + 2\cdot 0.05\right)s^2c^2\right] = 0.34E_1$$
$$\left(\overline{Q}_{16}\right)_{-45^\circ} = E_1\left[\left(1.009 - 0.1009 - 2\cdot 0.05\right)c^3s - \left(0.1009 - 0.03027 - 2\cdot 0.05\right)cs^3\right] = -0.23E_1$$
$$\left(\overline{Q}_{26}\right)_{-45^\circ} = E_1\left[\left(1.009 - 0.1009 - 2\cdot 0.05\right)cs^3 - \left(0.1009 - 0.03027 - 2\cdot 0.05\right)c^3s\right] = -0.23E_1$$
$$\left(\overline{Q}_{66}\right)_{-45^\circ} = E_1\left[\left(1.009 + 0.1009 - 2\cdot 0.03027 - 2\cdot 0.05\right)s^2c^2 + 0.05\left(s^4 + c^4\right)\right] = 0.26E_1$$

where $c = \cos -45,\ s = \sin -45$

$$\left[\overline{Q}\right]_{-45^\circ} = E_1\begin{bmatrix} 0.34 & 0.24 & -0.23 \\ 0.24 & 0.34 & -0.23 \\ -0.23 & -0.23 & 0.26 \end{bmatrix}$$

The transformed lamina stiffness matrix $[\overline{Q}]$ for the +45° ply are given by

$$\left[\overline{Q}\right]_{45^\circ} = E_1\begin{bmatrix} 0.34 & 0.24 & 0.23 \\ 0.24 & 0.34 & 0.23 \\ 0.23 & 0.23 & 0.26 \end{bmatrix}$$

Calculate the laminate extensional stiffness matrix $[A]$

$$A_{ij} = \sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k\left(z_k - z_{k-1}\right)$$

$$A_{11} = 2t\left[\cdot\left(\overline{Q}_{11}\right)_{-45} + \left(\overline{Q}_{11}\right)_{45}\right] =$$
$$= 2t\cdot\left(0.34 + 0.34\right)E_1$$
$$= 1.37\ tE_1$$

Similarly

$A_{12} = 2t\cdot\left(0.24 + 0.24\right)E_1 = 0.97\ tE_1,\ A_{22} = 2t\cdot\left(0.34 + 0.34\right)E_1 = 1.37\ tE_1$

$A_{16} = 2t\cdot\left(-0.23 + 0.23\right)E_1 = 0,\ A_{26} = 2t\cdot\left(-0.23 + 0.23\right)E_1 = 0$

$A_{66} = 2t\cdot\left(0.26 + 0.26\right)E_1 = 1.05\ tE_1$

$$[A] = \begin{bmatrix} 1.37 & 0.97 & 0 \\ 0.97 & 1.37 & 0 \\ 0 & 0 & 1.05 \end{bmatrix} tE_1,\ [B] = 0 \text{ (the laminate is symmetric)}$$

[This question was generally done well.]

2(a) Good corrosion resistance

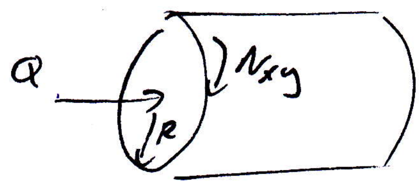
Low thermal expansion

Perhaps high E/ρ avoids whirl problems

Cost not so critical

Low weight not so critical as not for transport

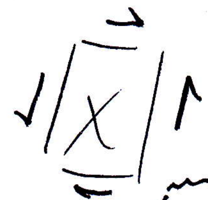
(b) (i)



$N_{xy} . 2\pi R . R = Q$

$\gamma = \tau / G$

$\tau . t = N_{xy}$



At failure $\gamma = e_{LT}$

This step commonly missed

From Table 1, $e_{LT} = 0.0005$ *

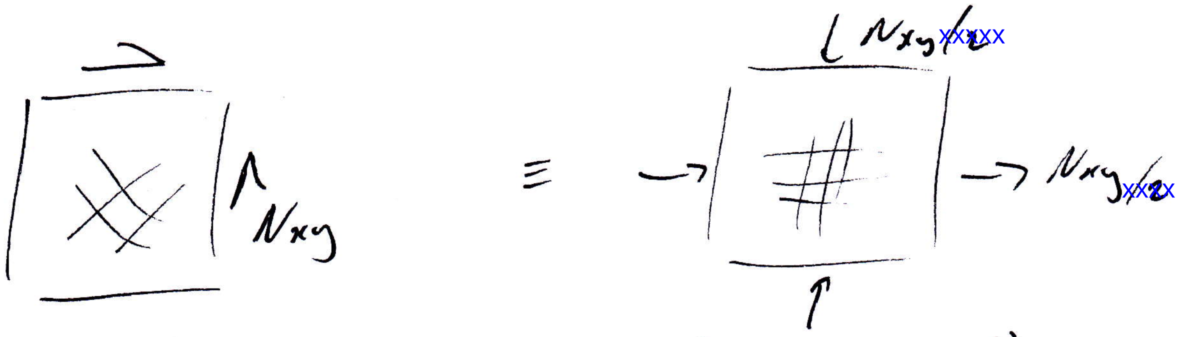
$G \simeq 35 GPa$ (this is appropriate for the $\pm 45°$ plies)

$\Rightarrow Q = G e_{LT} . t . 2\pi R^2$

$= 35 \times 10^9 \times 0.005 \times 4 \times 10^{-3} \times 2\pi \times (50 \times 10^{-3})^2 = 11.0 \text{ kNm}$

* NB This model assumes $\pm 45°$ and $0°$ plies, but failure is expected due to transverse tension of $\pm 45°$ plies, so a reasonable model.

2 (b)(i) A matrix → ε in laminate → ε in ply → σ in ply



Either

$$\begin{pmatrix} 0 \\ 0 \\ N_{xy} \end{pmatrix} = \begin{pmatrix} - & - & 0 \\ - & - & 0 \\ 0 & 0 & A_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

⇒ only need A_{66}

$Q_{11} = E_1/((1-\nu_{12}\nu_{21}) = 138.8\ GPa$

$Q_{22} = 9.05\ GPa$

$Q_{12} = 2.72\ GPa$

$Q_{66} = 6.9\ GPa$

$\bar{Q}_{66} = 35.6\ GPa$

$A_{66} = \underset{\text{thickness}}{4\times10^{-3}} \times 35.6\times10^{9}\ Nm^{-1}$

$\gamma_{xy} = N_{xy}/A_{66} = N_{xy} . 7.02\times10^{-9}\ mN^{-1}$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = T^{-T} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \frac{\gamma_{xy}}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

for $-45°$ ply, assumed critical as transverse tension.

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} 138.8 & 2.72 \\ 2.72 & 9.05 \end{pmatrix} \times 10^{9} \times \begin{pmatrix} -1 \\ 1 \end{pmatrix} \times \frac{7.02\times10^{-9}}{2}\ m^{-1}$$

$$= \begin{pmatrix} -478 \\ 22 \end{pmatrix} N_{xy}\ m^{-1}$$

Or (easier, I think)

$$\begin{pmatrix} N_{xy}/\cancel{2} \\ -N_{xy}/\cancel{2} \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & - \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

$$[A] = 2\times10^{-3} \begin{pmatrix} 138.3+9.05 & 2.72+2.72 & - \\ 5.63 & 147.95 & - \\ - & - & - \end{pmatrix} \times 10^{9}\ Nm^{-1}$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} = \frac{N_{xy}}{\cancel{2}\times10^{6}} \underbrace{\begin{pmatrix} 295.7 & -10.88 \\ 10.88 & 295.7 \end{pmatrix} \frac{1}{\Delta}}_{[A^{-1}]:\ \Delta = 87{,}320\ (GPa^2)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} mN^{-1}$$

$$= N_{xy} . 3.52\times10^{-9} \begin{pmatrix} 1 \\ -1 \end{pmatrix} mN^{-1}$$

For ply with transverse tension:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \underset{[Q]}{\begin{pmatrix} 138.8 & 2.72 \\ 2.72 & 9.05 \end{pmatrix}} \times 10^{9} . N_{xy} . 3.52\times10^{-9} \times \begin{pmatrix} -1 \\ 1 \end{pmatrix} m^{-1}$$

$$= \begin{pmatrix} -479 \\ 22 \end{pmatrix} N_{xy}\ m^{-1}$$

$\boxed{\text{Quite a lot of time was wasted calculating unnecessary elastic constants}}$

2 (b) (ii)

Using Tsai–Hill, still assuming ply with transverse tension is critical

differences between the two plies in failure seldom mentioned

$$\frac{\sigma_1^2}{S_L^2} - \frac{\sigma_1 \sigma_2}{S_L^2} + \frac{\sigma_2^2}{S_T^2} = 1$$

$$\left(\frac{479}{1172}\right)^2 - \frac{479 \cdot (-22)}{1172^2} + \left(\frac{22}{48 \cdot 3}\right)^2 = \frac{1}{N_{xy}^2} (MPa)^2 m^2$$

$$\Rightarrow N_{xy} = 1.62 \ MN\,m^{-1}$$

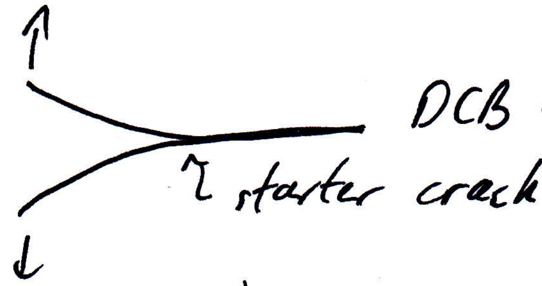
$$Q = N_{xy} \cdot 2\pi R^2$$
$$= 25.6 \ kNm$$

(c) Results differ by $\times 2.3$, with the strain allowable approach being very conservative. Probably the shear allowable e_{LT} is to conservative but also note that the absence of the assumed 0° plies may be affecting that calculation.

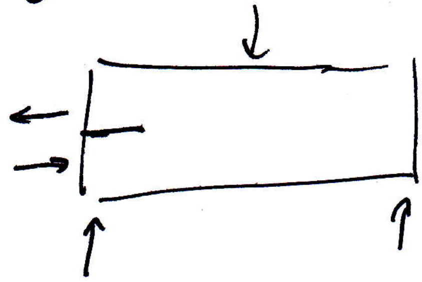
[In fact for $Q = 25.6$ kNm, γ_{xy} for the laminate is ~~only~~ 1.1%, so significantly higher than the allowable of 0.5%]

3 (a)

Need to measure toughness at inter-laminar interface.

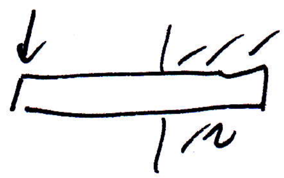


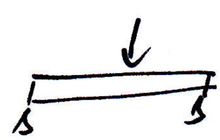
DCB → mode I toughness



Mode II and mixed mode using end-notched flexure

In addition characterise, in a more qualitative way, delamination resistance using impact and compression-after-impact tests





3 (b) The laminate compliance S relates ε and σ via $\varepsilon = S\sigma$. $\boxed{\text{This was often missed}}$

This is expressed in laminate plate theory via

$$N = A\varepsilon \quad \text{where} \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = {}^{1/t}\ t \begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix}$$

so $S = \frac{1}{t} A^{-1}$ tA^{-1}

Here $Q_{11} = 39.6$ GPa, $Q_{22} = 8.6$ GPa

$Q_{12} = 2.19$ GPa $Q_{16} = Q_{26} = 0$, $Q_{66} = 4.1$ GPa

$$A = 0.5\,\text{mm} \begin{pmatrix} 39.6 + 8.6 & 2.19 \times 2 & 0 \\ 2.19 \times 2 & 39.6 + 8.6 & 0 \\ 0 & 0 & 8.2 \end{pmatrix} \text{GPa}$$

$$= \begin{pmatrix} 24 & 2.19 & 0 \\ 2.19 & 24 & 0 \\ 0 & 0 & 4.1 \end{pmatrix} \times 10^{6}\ \text{N m}^{-1}$$

tA{-1} 10^{-3}

$$S = \frac{1}{t} A^{-1} = \frac{1}{10^{-3}\,\text{m}} \begin{pmatrix} 24/\Delta & -2.19/\Delta & 0 \\ -2.19/\Delta & 24/\Delta & 0 \\ 0 & 0 & 1/4.1 \end{pmatrix} \times 10^{-6}\ \text{m N}^{-1}$$

$$\Delta = 571.2\ \text{N}^2\text{m}^{-2}$$

$$= \begin{pmatrix} 0.042 & -0.0038 & 0 \\ -0.0038 & 0.042 & 0 \\ 0 & 0 & 0.24 \end{pmatrix} \text{GPa}^{-1}$$

3 (c) Assume $K_I = 1 \times \sigma\sqrt{\pi a}$ ← assumed Y

$K_{II} = 1 \times \tau\sqrt{\pi a}$ where a is half-length $= 25\,mm$

– this is an approximation for an isotropic plate of infinite size which will be reasonable

Now find effective modulus for fracture mechanics

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{\frac{1}{2}}\left\{\left(\frac{S_{22}}{S_{11}}\right)^{\frac{1}{2}}\left(1 + \frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right\}^{\frac{1}{2}}$$

$$= \frac{0.062}{\sqrt{2}}\left(1 \times \left(1 + \frac{-2\times 0.0038 + 0.26}{2\times 0.062}\right)\right)^{\frac{1}{2}} = 0.0576\ GPa^{-1}$$

$= \dfrac{1}{17.3\ GPa}$ – seems reasonable

$E'_A = E'_B$ since $S_{11} = S_{22}$

(i) $G_I = K_I^2/E'_B \Rightarrow \sigma_c = \dfrac{K_I}{\sqrt{\pi a}} = \sqrt{\dfrac{G_{Ic}E'_B}{\pi a}} = \sqrt{\dfrac{20\times10^3 \times 17.3\times10^9}{\pi\times 25\times10^{-3}}}$ $= 66\ MPa$

(ii) $G_{II} = K_{II}^2/E'_A \Rightarrow \tau_c = \sqrt{\dfrac{60\times10^3 \times 17.3\times10^9}{\pi\times25\times10^{-3}}} = 96\ MPa$

(iii) $G = G_I + G_{II} = \dfrac{\pi a}{E'_A}\left(\sigma^2 + \left(\dfrac{\sigma}{2}\right)^2\right)$

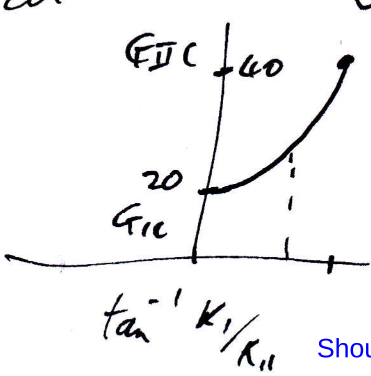
Need to make some assumption about G_c – assume $= 30\ kJ/m^2$

$\Rightarrow \sigma = 2\tau = \sqrt{\dfrac{30\times10^3 \times 17.3\times10^9}{25\times10^{-3}\times\pi\times 5/4}}$

$\boxed{\text{not } G_c = G_{Ic} + G_{IIc}}$

$= 73\ MPa$

Expect



Should be tan^{-1}(KII/KI)

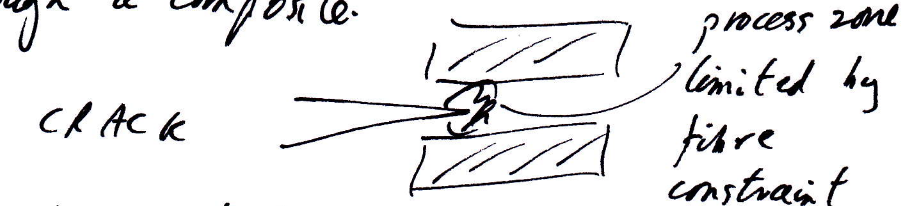
$\dfrac{K_I}{K_{II}} = 2,\ \psi = 63°$

perhaps $G_c \approx 30\ kJ/m^2$ — $\boxed{\text{other assumptions accepted if explained}}$

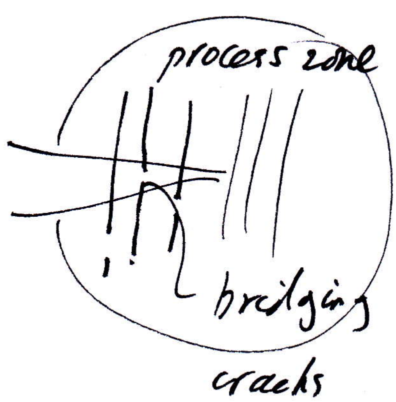
psi=27 degrees, GC=22 approx (20+20/9)
sigma=62 MPa

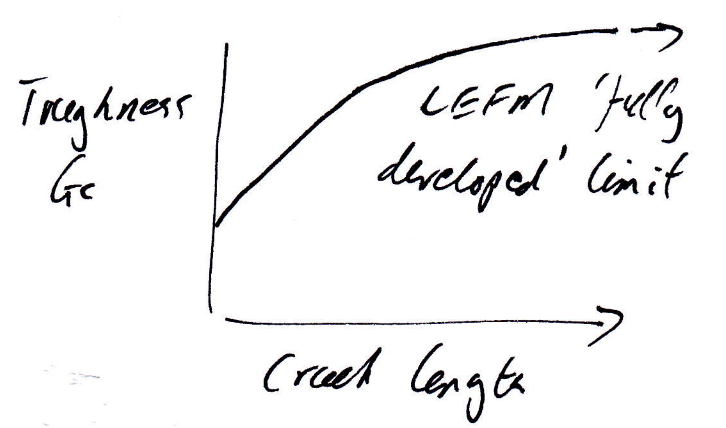
3(d) LEFM is applicable when the process zone is small compared to crack or specimen dimensions.

This isn't valid in general when assessing crack growth at the micro scale between fibres when the process zone is large compared with the gaps between fibres. This leads to a difference between crack propagation in neat matrix and through a composite.



At a larger scale, fibre bridging provides a mechanism at the tip of cracks which affect toughness. Such damage zones in laminates are of the order of mm in length so that the effective toughness is not a material property independent of crack length. This leads to a so-called R-curve (R = resistance) with toughness increasing with crack length.





4. (a) **Structural** considerations: this handle will be very long and thin, and is likely to be deflection limited (as the strains associated with quite large curvatures will be within the strain allowables of composites). We expect the structure to be a thin-walled tube, to be able to maximize the structural efficiency. In principle it would be advantageous to taper the handle towards the tip. But depending on the processing route this may be less easy to do.

Weight will be important to allow people to be able to manoeuvre the brush. **Cost** will also be relevant, but perhaps an effective piece of equipment will pay for itself quite quickly, in comparison with the time associated with ladders. A trade-off calculation will need to be done to assess the cost advantage of a lighter design.

The above general considerations suggest that a **CFRP solution** might be appropriate. Bearing in mind the need for high bending stiffness, the lay-up will be predominantly with fibres running along the axis of the tube. This seems to be appropriate to pultrusion, with some off-axis material (perhaps woven) included in the pultruded layup to provide protection against splitting or impact, and torsional stiffness. The relatively low cost of this route is also attractive. (See the notes for a detailed description of pultrusion).

Although in principle the handle could be made in one piece, perhaps it would be worth having sections of reducing diameter joined in some way to allow for the **tapering** suggested by the structural consideration. (This could not be achieved by pultrusion but could be achieved by some form of layup, perhaps hand or automated tape layup.) Perhaps inserts with step-downs in diameter or concentric tubes as per extendable aerials could work. As well as these joints, joining at the brush end will be important, perhaps again using a fitting to avoid premature failure.

Testing will be important for product quality purposes, including rough handling and impact events, though the component is not as safety critical as say automotive or aerospace applications.

[In general this part was well answered, though comments in all areas were needed to score high marks.]]

(b) Woven materials have fibres which do not run parallel to the loading direction. This means that their contribution to the stiffness is less. Moreover compressive strength is sensitive to fibre misalignment or load misalignment, as this induces shear in the matrix causing microbuckling. This is a cause of reduced compressive strength. [Not well answered]

(c) Barely visible damage is caused by relatively low impact events such as dropping of tools or stones impacting the structure. A key problem is that, by definition, this damage is difficult to see so may be present without the operator knowing it. So this means that the structure needs to be designed to cope with such damage. Often the damage is in the form of sub-surface delamination, which will be difficult to see but may lead to premature failure, either due to fatigue growth or due to buckling of the delaminated region under a compressive loading.

(d) Such tanks are quite large (several metres in length) so the relatively low weight of GFRP will make transport and installation in the remote locations easier. While the deadweight loading of covering soil will not be negligible requiring reasonable structural properties. In addition GFRP is relatively low cost for this commodity item and can be formed into the required curved shapes easily. Finally GFRP has good corrosion resistance against water and other chemicals in the sewage. All of the above indicate GFRP as a good choice.